

# UNIFYING THE MODEL THEORY OF FIRST-ORDER AND SECOND-ORDER ARITHMETIC VIA $\text{WKL}_0^*$

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ABSTRACT. We develop machinery to make the Arithmetized Completeness Theorem more effective in the study of many models of  $\text{I}\Delta_0 + \text{B}\Sigma_1 + \text{exp}$ , including all countable ones, by passing on to the conservative extension  $\text{WKL}_0^*$  of  $\text{I}\Delta_0 + \text{B}\Sigma_1 + \text{exp}$ . Our detailed study of the model theory of  $\text{WKL}_0^*$  leads to the simplification and improvement of many results in the model theory of Peano arithmetic and its fragments pertaining to the construction of various types of end extensions and initial segments.

## 1. INTRODUCTION

In this paper we develop the model theory of the subsystem  $\text{WKL}_0^*$  of the well-known fragment  $\text{WKL}_0$  of second-order arithmetic with an eye towards building a versatile framework that serves as a platform to apply appropriate forms of the Arithmetized Completeness Theorem to unify, extend, and refine a rather large body of work concerning models of both first-order and second-order arithmetic.

The subsystem  $\text{WKL}_0^*$  of second-order arithmetic was first introduced and studied by Simpson and Smith [60] who showed, among other things, that the first-order consequences of  $\text{WKL}_0^*$  coincide with the fragment of Peano arithmetic axiomatized by  $\Delta_0$  induction ( $\text{I}\Delta_0$ ) plus  $\Sigma_1$  collection ( $\text{B}\Sigma_1$ ) and the totality of exponentiation ( $\text{exp}$ ); this was accomplished by a forcing argument which demonstrated that every countable model of  $\text{I}\Delta_0 + \text{B}\Sigma_1 + \text{exp}$  has an expansion to a model of  $\text{WKL}_0^*$ . Here we show (in Corollary 4.2) that models of  $\text{I}\Delta_0 + \text{B}\Sigma_1 + \text{exp}$  that have an expansion to a model of  $\text{WKL}_0^*$  are precisely those that have a proper end extension to a model of  $\text{I}\Delta_0$ . This result serves as a useful bridge between the work by the Paris–Wilkie school in first-order arithmetic, and the work by the Friedman–Simpson school in second-order arithmetic. For example, it shows that the aforementioned Simpson–Smith expansion result can be derived without the use of forcing from the existence of proper end extensions satisfying  $\text{I}\Delta_0$  of countable models of  $\text{I}\Delta_0 + \text{B}\Sigma_1 + \text{exp}$ ; this latter fact is due to Wilkie and Paris [74]. Our work also illustrates ways in which the links between collection schemes and subsystems of second-order arithmetic can be exploited, in the context where exponentiation is assumed to be total.

We begin in Section 2 by setting up notation and surveying the connections between  $\text{I}\Delta_0 + \text{B}\Sigma_1 + \text{exp}$  and  $\text{WKL}_0^*$ . Then in Section 3 we employ tableau deductions (or, equivalently, cut-free deductions) to craft an appropriate form of the Arithmetized Completeness Theorem that can be accommodated in  $\text{WKL}_0^*$ . The machinery developed in Section 3 is put to work in Section 4 to provide a coherent, polished view of a diverse collection of results concerning end extensions of models of arithmetic. For example, Theorem 4.6 refines the Wilkie–Paris theorem mentioned in

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the previous paragraph on the existence of end extensions of countable models of  $\text{I}\Delta_0 + \text{B}\Sigma_1 + \text{exp}$ ; Theorem 4.12 provides an alternative proof of a recent theorem of Schmerl's [55] that generalizes the Mac Dowell–Specker theorem by characterizing families of subsets of a given model of Peano arithmetic (PA) that can be coded in an elementary end extension; and Theorem 4.14 improves a characterization by Wilkie [72] of the possible theories satisfied in an end extension of a model of PA. In Section 5 we complement the ‘outward’ outlook of Section 4 by turning ‘inward’ towards the rich landscape of initial segments of a given nonstandard model of arithmetic. For example, Theorem 5.4 establishes a Tanaka-style self-embedding theorem [68] for models of  $\text{WKL}_0^*$ ; Theorem 5.6 refines a result of Lessan's [43] concerning complete extensions of PA that can be realized as the theory of arbitrarily high recursively saturated initial segments of a model of arithmetic; and Theorem 5.10 extends a theorem of Kaye's [35] that characterizes theories of arithmetic that can be satisfied by an initial segment of a model of arithmetic below a prescribed nonstandard element. We conclude the paper with some open questions in Section 6.

## 2. PRELIMINARIES

All unexplained concepts and notations can be found in Hájek–Pudlák [32], Kaye [35], Kossak–Schmerl [41], or Simpson [59]. All our languages include  $\mathcal{L}_A = \{0, 1, +, \times, \leq\}$ . All the  $\text{B}\Sigma_n$ 's are assumed to include  $\text{I}\Delta_0$ , so that we will not need to write  $\text{I}\Delta_0 + \text{B}\Sigma_1$  any more. Similarly, all the  $\text{B}\Sigma_n^0$ 's include  $\text{I}\Sigma_0^0$ . Unless otherwise stated, definability means parametric definability. If  $M$  is a structure, then  $\text{Def}(M)$  denotes the set of all definable subsets of  $M$ .

It is well known [32, Section V.3(c)] that in  $\text{I}\Delta_0$ , there is a  $\Delta_0$  formula expressing  $y = 2^x$ . By  $m \in_{\text{Ack}} d$ , we denote a  $\Delta_0$  formula which expresses ‘the  $m$ th digit in the binary expansion of  $d$  is 1’. If  $M$  is a proper initial segment of a model  $K \models \text{I}\Delta_0$ , then

$$\text{SSy}_M(K) = \{\{m \in M : K \models m \in_{\text{Ack}} d\} : d \in K\}.$$

**Proposition 2.1** (folklore). Let  $a \in M \models \text{I}\Delta_0 + \exists y (y = 2^a)$ . Then

$$M \models \forall \bar{z} \exists d \forall x < a (\varphi(x, \bar{z}) \leftrightarrow x \in_{\text{Ack}} d)$$

for every  $\Delta_0$  formula  $\varphi(x, \bar{z})$ .

*Proof.* Given  $\bar{z} \in M$ , show by induction on  $a'$  that

$$M \models \forall a' \leq a \exists d < b \forall x < a' (\varphi(x, \bar{z}) \leftrightarrow x \in_{\text{Ack}} d),$$

where  $b$  is some fixed element of  $M$  bigger than  $2^a$ . See Lemma IV.2.12 in [32] for the details.  $\square$

Over  $\text{I}\Delta_0$ , the usual apparatus for coding pairs and sequences works reasonably smoothly. We denote the code of an ordered pair  $(x, y)$  by  $\langle x, y \rangle$ . This notation extends to codes of  $n$ -tuples for any fixed  $n \in \mathbb{N}$ . By  $[x_1, x_2, \dots, x_\ell]$ , we denote a code for the sequence  $(x_1, x_2, \dots, x_\ell)$  of possibly nonstandard length.

All arithmetic operations can naturally be viewed as relations. More specifically, if  $\mathcal{L}$  is a language, then we denote by  $\mathcal{L}^{\text{rel}}$  the language obtained from  $\mathcal{L}$  by replacing each function symbol  $f$  with a relation symbol  $f^{\text{rel}}$  whose arity is one more than that of  $f$  and whose intended interpretation is the graph of  $f$ . When there is no risk of ambiguity, we will sometimes identify  $\mathcal{L}, f$  with their relational counterparts  $\mathcal{L}^{\text{rel}}, f^{\text{rel}}$ . With this point of view, every nonempty subset of a model of arithmetic gives rise to a substructure. Abusing notation, if  $s \in M \models \text{I}\Delta_0$ , then we use again the letter  $s$  to refer to  $\{x \in M : x < s\}$  enriched with the arithmetic operations inherited from  $M$ .

**Theorem 2.2** (Lessan). There is a  $\Delta_0$  formula  $\text{Sat}_0(x, y, z)$  with the following property:

If  $s, t \in M \models \text{I}\Delta_0$  such that  $t > 2^{s^{\aleph_1}}$ , then for all  $\mathcal{L}_A^{\text{rel}}$  formulas  $\theta(\bar{x})$  and for all  $\bar{x} < s$ ,

$$s \models \theta(\bar{x}) \iff M \models \text{Sat}_0(\ulcorner \theta \urcorner, [\bar{x}], t).$$

*Proof.* Lessan [43, Lemma 4.1.4] stated this for  $M \models \text{PA}$  instead of  $M \models \text{I}\Delta_0$ , but the proof is the same; cf. Section V.5(b) in [32].  $\square$

**Corollary 2.3** (Lessan). If  $s \in M \models \text{I}\Delta_0$  such that  $2^{s^{\aleph_1}} \subsetneq_e M$ , then  $s$  is recursively saturated.

*Proof.* This follows from Theorem 2.2 by a routine overspill argument [43, Proposition 4.1.5].  $\square$

As is well known, the addition of the  $\Sigma_1$  collection scheme preserves the  $\Pi_2$  consequences to a great variety of theories.

**Proposition 2.4** (folklore).  $\text{B}\Sigma_1 + \text{exp} + \varphi(c)$  is  $\Pi_2$ -conservative over  $\text{I}\Delta_0 + \text{exp} + \varphi(c)$  for all  $\varphi \in \Sigma_2$ , where  $c$  is a new constant symbol.

*Proof.* One finds a proof in Corollary 4.1 of Beklemishev [10], for example. We present here another one. Let  $\theta(y, z)$  be a  $\Sigma_1$  formula such that  $\text{I}\Delta_0 + \text{exp} + \varphi(c) + \neg \forall y \theta(y, c)$  is consistent, and suppose  $\varphi(c)$  is  $\exists x \psi(x, c)$ , where  $\psi \in \Pi_1$ . Take any recursively saturated  $K \models \text{I}\Delta_0 + \text{exp} + \exists x \psi(x, c) + \neg \forall y \theta(y, c)$ . Find  $a, b \in K$  such that  $K \models \psi(a, c) \wedge \neg \theta(b, c)$ . Define  $M = \sup\{\text{exp}^{(n)}(a + b + c) \in K : n \in \mathbb{N}\}$ , where  $\text{exp}^{(n)}$  denotes the  $n$ -fold iteration of  $\text{exp}: x \mapsto 2^x$ . We know  $M \neq K$  by recursive saturation. So an overspill (as in Theorem 10.5 in [35], for example) shows  $M \models \text{B}\Sigma_1 + \text{exp}$ . Since  $\psi, \neg \theta \in \Pi_1$ , we have  $M \models \psi(a, c) \wedge \neg \theta(b, c)$ , as required.  $\square$

See Buss [14] for similar conservation results about weaker theories, which were proved using a version of model theoretic forcing in countable recursively saturated models (as an alternative to a proof theoretic argument). See also the end of Section 3 for other related results.

Define

$$\text{RCA}_0^* = \text{I}\Sigma_0^0 + \text{exp} + \Delta_1^0\text{-CA} \quad \text{and} \quad \text{WKL}_0^* = \text{RCA}_0^* + \text{WKL},$$

where WKL is a sentence in the language of second-order arithmetic which expresses ‘every 0–1 tree with a node at every level has a cofinal path’. We know from Simpson–Smith [60, Theorem 4.3] that  $\text{RCA}_0^* \vdash \text{B}\Sigma_1^0$ . Some results in this paper generalize to higher levels of the arithmetic hierarchy. These generalizations rely on the theorem below, which shows that if  $n \in \mathbb{N}$  and  $M \models \text{B}\Sigma_{n+1} + \text{exp}$ , then  $(M, \Delta_{n+1}\text{-Def}(M)) \models \text{RCA}_0^*$ .

**Theorem 2.5** (folklore).  $\text{B}\Sigma_{n+1} \vdash \text{I}\Delta_{n+1}$  for all  $n \in \mathbb{N}$ .

*Proof.* See Theorem I.2.5(2) in [32].  $\square$

As Slaman [62] showed, the converse to Theorem 2.5 holds if one adds  $\text{exp}$  to the theories. We do not need this extra information in the present paper.

It has been known since Scott [56] that the Weak König Lemma goes hand in hand with end extensions. In our context, this correspondence can be formulated as follows.

**Proposition 2.6** (folklore). If  $M \models \text{I}\Delta_0 + \text{exp}$  and  $K \models \text{I}\Delta_0$  properly end extending  $M$ , then  $(M, \text{SSy}_M(K)) \models \text{WKL}_0^*$ .

*Proof.* By  $\Delta_0$  overspill, one sees  $K \models \exists y (y = 2^a)$  for some  $a \in K \setminus M$ . So we can follow Exercise 13.1 in [35] using Proposition 2.1; cf. Theorem 4.8 in [60] and Lemma IX.3.11 in [59].  $\square$

We will see partial converses to the proposition above in Proposition 4.1 and Theorem 4.6. It is proved in the Wilkie–Paris paper [74] that every countable  $M \models \text{B}\Sigma_1 + \text{exp}$  has a proper end extension  $K \models \text{I}\Delta_0$ ; cf. Dimitracopoulos–Paris [22, Theorem 1.8]. So Proposition 2.6 applies to all such models. Adamowicz [1] generalized this Wilkie–Paris end-extension theorem to all  $M \models \text{B}\Sigma_1 + \text{exp}$  of *countable cofinality*, i.e., all those  $M \models \text{B}\Sigma_1 + \text{exp}$  which possess a countable cofinal subset. The sentence  $\Omega_k$  below expresses the totality of the function  $\omega_k$  over  $\text{I}\Delta_0$ , where

$$\omega_0(x) = x \quad \text{and} \quad \omega_{k+1}(x) = x^{\omega_k(\log x)}$$

for all  $k \in \mathbb{N}$ .

**Theorem 2.7** (Adamowicz). Let  $M \models \text{B}\Sigma_1 + \text{exp}$  of countable cofinality. Then for every  $k \in \mathbb{N}$ , there exists a proper end extension  $K \supseteq_e M$  satisfying  $\text{B}\Sigma_1 + \Omega_k$ .

*Proof.* This is the corollary on page 144 of [1].  $\square$

**Corollary 2.8.** Every  $M \models \text{B}\Sigma_1 + \text{exp}$  of countable cofinality expands to  $(M, \mathcal{X}) \models \text{WKL}_0^*$ .  $\square$

A simple overspill argument shows that having a proper end extension satisfying  $\text{I}\Delta_0$  is equivalent to having a proper end extension satisfying  $\text{B}\Sigma_1 + \Omega_k$  for a model of  $\text{I}\Delta_0 + \text{exp}$  and  $k \in \mathbb{N}$ . For countable models, we can additionally squeeze out some recursive saturation.

**Theorem 2.9** (essentially Kaye). Let  $M$  be a countable model of  $\text{I}\Delta_0$  and  $a, r, s \in M$  such that  $a < r < s$  and  $s$  is recursively saturated. Let  $\Phi(z)$  be a set of  $\mathcal{L}_A$  formulas in  $\text{SSy}(M)$ . The following are equivalent.

- (a) There is  $I \subseteq_e M$  such that  $I \models \Phi(a)$  and  $r \in I < s$ .
- (b) There is a recursively saturated  $I \subseteq_e M$  such that  $I \models \Phi(a)$  and  $r \in I < s$ .

*Proof.* Follow the indicator argument in Section 14.2 of [35], or see Theorem 5.10 below for a direct proof with a similar flavor.  $\square$

**Corollary 2.10.** Let  $M \models \text{I}\Delta_0 + \text{exp}$  and  $k \in \mathbb{N}$ . For every countable  $K \models \text{I}\Delta_0$  properly end extending  $M$ , there exists a recursively saturated  $K' \models \text{B}\Sigma_1 + \Omega_k$  such that  $M \subsetneq_e K' \subsetneq_e K$ .

*Proof.* Overspill gives an interval  $[r, s] \subseteq K \setminus M$  such that

$$r \in \omega_{k+1}^{\mathbb{N}}(r) = \sup\{\omega_{k+1}^{(i)}(r) : i \in \mathbb{N}\} < s \in 2^{s^{\mathbb{N}}} \subsetneq_e K.$$

So by Corollary 2.3 and Theorem 2.9, one finds a recursively saturated  $K' \models \text{B}\Sigma_1 + \Omega_k$  inside  $[r, s]$ .  $\square$

As mentioned in the introduction, many arguments in this paper involve expanding a model of  $\text{B}\Sigma_1 + \text{exp}$  to a model of  $\text{WKL}_0^*$ . We call models of  $\text{B}\Sigma_1 + \text{exp}$  that admit such expansions  $\text{WKL}_0^*$ -*expandable*. For example, with this terminology, Corollary 2.8 can be rephrased as saying every model of  $\text{B}\Sigma_1 + \text{exp}$  of countable cofinality is  $\text{WKL}_0^*$ -expandable. Resplendent models of  $\text{B}\Sigma_1 + \text{exp}$  are clearly  $\text{WKL}_0^*$ -expandable, but so are all models of  $\text{I}\Sigma_1$  — see the end of this section.

For notational simplicity, if  $\bar{S} \subseteq M \models \text{I}\Delta_0$ , then we write  $(M, \bar{S})$  for the structure  $(M, \{\bar{S}\})$ . A set  $\mathcal{X}_0$  is said to  $\Delta_1^0$ -*generate* a model  $(M, \mathcal{X}) \models \text{RCA}_0^*$  if  $\mathcal{X}_0 \subseteq \mathcal{X}$  and every  $(M, \mathcal{Y}) \models \text{RCA}_0^*$  with  $\mathcal{X}_0 \subseteq \mathcal{Y}$  must include  $(M, \mathcal{X})$  as a substructure. Equivalently, a model  $(M, \mathcal{X}) \models \text{RCA}_0^*$  is  $\Delta_1^0$ -generated by a subset

$\mathcal{X}_0 \subseteq \mathcal{X}$  if and only if every element of  $\mathcal{X}$  is  $\Delta_1^0$ -definable in  $(M, \bar{S})$  for some  $\bar{S} \in \mathcal{X}_0$  as a subset of  $M$ .

A model  $(M, \mathcal{X}) \models \text{RCA}_0^*$  is *countably  $\Delta_1^0$ -generated*, or simply *countably generated*, if there is a countable subset of  $\mathcal{X}$  that  $\Delta_1^0$ -generates  $(M, \mathcal{X})$ . Similarly, the model  $(M, \mathcal{X})$  is *finitely  $\Delta_1^0$ -generated*, or *finitely generated*, if it is  $\Delta_1^0$ -generated by a finite subset of  $\mathcal{X}$ . Using the usual coding of sets in models of  $\text{RCA}_0^*$ , one sees that finitely generated models of  $\text{RCA}_0^*$  can actually be generated by singletons. Therefore, such models are also called *principal* models in the literature [61]. Countable models of  $\text{RCA}_0^*$  are clearly countably generated, and it is not hard to find principal models of  $\text{RCA}_0^*$  which are not countable.

Upon closer inspection, one sees that the proof of Theorem 2.7 actually shows every  $(M, S) \models \text{BS}\Sigma_1^0 + \text{exp}$  of countable cofinality has a proper end extension  $K \models \text{ID}_0$  in which  $S \in \text{SSy}_M(K)$ . Consequently, every principal model  $(M, \mathcal{X}) \models \text{RCA}_0^*$  of countable cofinality has an expansion  $(M, \mathcal{Y}) \models \text{WKL}_0^*$  by Proposition 2.6. As shown independently by Paris [48, Theorem 2] and Hájek [31, Theorem 3.13(a)], the same is true for higher levels of the collection hierarchy, even when the countable cofinality condition is dropped. Paris’s proof uses a ‘mild refinement’ of the Arithmetized Completeness Theorem to construct a suitable end extension. Hájek’s proof consists of an  $\omega$ -interpretation based on ‘very low’ sets.

For countably generated models, an extra countability assumption seems to be needed.

**Theorem 2.11** (Simpson–Smith, Avigad). Every countable  $(M, \mathcal{X}) \models \text{BS}\Sigma_{n+1}^0 + \text{exp}$ , where  $n \in \mathbb{N}$ , can be extended to  $(M, \mathcal{Y}) \models \text{WKL}_0^* + \text{BS}\Sigma_{n+1}^0$ .

*Proof.* The case when  $n = 0$  was first proved by Simpson and Smith [60, Theorem 4.6]. The general case is Theorem 2.7 in Avigad [9]. See also Belanger [11].  $\square$

We end this preliminary section with some analogous results for induction schemes, which, strictly speaking, are outside the scope of the present paper.

**Theorem 2.12** (Harrington, Avigad). Every countable  $(M, \mathcal{X}) \models \text{I}\Sigma_{n+1}^0$ , where  $n \in \mathbb{N}$ , can be extended to  $(M, \mathcal{Y}) \models \text{WKL}_0^* + \text{I}\Sigma_{n+1}^0$ .

*Proof.* The case when  $n = 0$ , due to Harrington, appears as Theorem IX.2.1 in [59]. (A weaker result is announced independently in Ratajczyk [52].) The general case is Theorem 2.4 in Avigad [9]. See also Cholak–Jockusch–Slaman [16, comment on page 52] and Belanger [11].  $\square$

As in the case for collection schemes, Paris [48, Theorem 5] showed, using again his ‘mild refinement’ of the Arithmetized Completeness Theorem, that the countability assumption in Theorem 2.12 can be dropped if the model  $(M, \mathcal{X})$  is assumed to be principal *and*  $n \geq 1$ . He [48, second part of Problem 1] asked whether the same holds for uncountable models when  $n = 0$ . An affirmative answer follows from Theorem 3.1 of Hájek [31]. More generally, as Avigad [8] showed, Theorem 2.12 is actually true without the countability assumption when  $n = 0$ .

### 3. THE ARITHMETIZED COMPLETENESS THEOREM, REVISITED

Arithmetization of syntax is done as in Wilkie–Paris [73], except that we will take advantage of the language of second-order arithmetic. We often identify a formula with its Gödel number. Write  $\check{a}$  for the canonical term

$$0 + \underbrace{1 + 1 + \cdots + 1}_{a\text{-many } 1\text{'s}}.$$

Fix a suitable  $\Sigma_1^0$  formula  $\text{Prov}(T, \theta)$  with free variables  $T$  and  $\theta$  that expresses over  $\text{RCA}_0^*$  the provability of  $\theta$  from  $T$ . Let  $\text{Con}(T)$  abbreviate  $\neg \text{Prov}(T, \perp)$ ,

where  $\perp$  is the symbol for falsum. In the first-order setting, if  $T$  is definable, then  $\text{Prov}(T, \theta)$  and  $\text{Con}(T)$  also make sense within  $\text{I}\Delta_0 + \text{exp}$ .

The syntactical version of the Compactness Theorem is true within  $\text{RCA}_0^*$ , although the semantical version requires the Weak König Lemma [59, Theorem IV.3.3].

**Lemma 3.1.**  $\text{RCA}_0^*$  proves that if every bounded coded subset of a theory  $T$  is consistent, then  $T$  is consistent.

*Proof.* Let  $(M, \mathcal{X}) \models \text{RCA}_0^*$ . Pick  $T \in \mathcal{X}$  such that  $(M, \mathcal{X}) \models \neg \text{Con}(T)$ . Let  $p \in M$  which witnesses this. Extract the set  $t$  of all elements of  $T$  used in  $p$ . This  $t$  is a bounded coded subset of  $T$  by Proposition 2.1, and  $M \models \neg \text{Con}(t)$ .  $\square$

The usual proof of Lindenbaum's Lemma using the Weak König Lemma goes through in  $\text{WKL}_0^*$ .

**Lemma 3.2.**  $\text{WKL}_0^*$  proves that every consistent theory has a consistent complete extension.

*Proof sketch.* Work within  $\text{WKL}_0^*$ . Let  $T$  be a consistent theory. As in the proof of Lemma IV.3.2 in [59], use  $\Delta_1^0$  comprehension to create a tree whose cofinal branches correspond to the consistent complete extensions of  $T$ . Showing this tree has a node at every level amounts to showing

for every sequence  $\theta_1, \theta_2, \dots, \theta_\ell$  of formulas, there exists a coded function  $\sigma: \ell \rightarrow 2$  such that

$$T \cup \{\neg\theta_i : \sigma(i+1) = 0\} \cup \{\theta_i : \sigma(i+1) = 1\}$$

is consistent.

With  $\text{B}\Sigma_1^0 + \text{exp}$ , this is simply a matter of piecing deductions together using the cut rule. An application of WKL then gives us what we want.  $\square$

**Corollary 3.3.**  $\text{WKL}_0^*$  proves that every consistent theory can be extended to a consistent complete Henkinized theory.

*Proof.* In view of Lemma 3.2, it suffices to show that the addition of Henkin axioms preserves consistency. The usual syntactical argument works here as in the proof of Theorem IV.3.3 in [59]. Here are some details.

Work over  $\text{WKL}_0^*$ . Let  $T$  be a consistent theory. Consider the language  $\mathcal{H}$  obtained from the language of  $T$  by adding unboundedly many new constant symbols. Let  $c_0, c_1, c_2, \dots$  be an enumeration of these new constant symbols, and  $\psi_0(v), \psi_1(v), \psi_2(v), \dots$  be an enumeration of all  $\mathcal{H}$  formulas with exactly one free variable  $v$  such that  $c_j$  does not appear in  $\psi_i$  for any  $i, j$  with  $i < j$ . We will establish the consistency of

$$T \cup \{\exists v \psi_i(v) \rightarrow \psi_i(c_i) : i < \ell\}$$

for every  $\ell$ . An application of Lemma 3.1 will then finish the proof.

Fix  $\ell$ . As in the previous proof, find a coded  $\sigma: \ell \rightarrow 2$  such that

$$T \cup \{\neg\exists v \psi_i(v) : \sigma(i) = 0\} \cup \{\exists v \psi_i(v) : \sigma(i) = 1\}$$

is consistent. Our bookkeeping condition on the  $c_i$ 's and the  $\psi_i$ 's then enables us to apply existential instantiation to get also the consistency of

$$T \cup \{\neg\exists v \psi_i(v) : \sigma(i) = 0\} \cup \{\psi_i(c_i) : \sigma(i) = 1\}.$$

This implies what we want because one can easily deduce  $\exists v \psi_i(v) \rightarrow \psi_i(c_i)$  from either  $\neg\exists v \psi_i(v)$  or  $\psi_i(c_i)$  by disjunction introduction. In any of the usual deduction systems, the increase in size in these syntactical manipulations is at most polynomial in  $2^{\ell^2}$  and the deductions involved [32, Section V.3(f) and (g)]. Therefore, our background theory  $\text{WKL}_0^*$  is capable of performing them.  $\square$

Let  $T$  be a consistent complete Henkinized theory in a model of  $I\Sigma_0^0 + \text{exp}$ . Then the set of all sentences in  $T$  with *standard shapes*, i.e., those that can be obtained from a standard formula by substituting all or some of its free variables with constant symbols in  $M$ , is a consistent complete Henkinized theory when viewed externally. So a structure consisting entirely of realizations of Henkin constants can be read off directly from  $T$ . This structure, which we refer to as the *Henkin model of  $T$* , satisfies all sentences in  $T$  with standard shapes.

The Arithmetized Completeness Theorem is a powerful way of building models of arithmetic. See Smoryński [64] and McAloon [45], for example, for some classical applications. Corollary 3.3 gives the following version of the Arithmetized Completeness Theorem for models of  $\text{WKL}_0^*$ .

**Proposition 3.4.** Let  $(M, \mathcal{X}) \models \text{WKL}_0^*$  and  $T \in \mathcal{X}$ . If  $(M, \mathcal{X})$  believes  $T$  is a consistent theory that includes  $\text{PA}^-$ , then there is a proper end extension  $K \supseteq_e M$  with the following properties.

- (1) If  $\theta(\check{m}_1, \check{m}_2, \dots, \check{m}_k) \in T$  where  $\theta(\bar{x})$  has a standard shape, then  $K \models \theta(m_1, m_2, \dots, m_k)$ .
- (2) There are  $S \in \mathcal{X}$  and a surjection  $e: M \rightarrow K$  such that
 
$$\{(j, s, t) \in S : j \in \mathbb{N}\} = \{\ulcorner \theta^\ulcorner, [\check{m}], [\check{c}] \urcorner \in M : \theta \in \mathcal{L}_A \text{ and } K \models \theta(\check{m}, e(\check{c}))\}.$$
- (3) All definable cuts of  $K$  include  $M$  properly.
- (4)  $K$  is recursively saturated.

*Proof.* First, suppose  $M \neq \mathbb{N}$ . Let  $(\psi_\ell(v, \bar{z}_\ell))_{\ell \in M}$  be an enumeration of all formulas in the language of  $T$  in  $(M, \mathcal{X})$ . Consider

$$\hat{T} = T \cup \{\forall \bar{z}_\ell (\psi_\ell(0, \bar{z}_\ell) \wedge \forall v (\psi_\ell(v, \bar{z}_\ell) \rightarrow \psi_\ell(v+1, \bar{z}_\ell)) \rightarrow \psi_\ell(d_\ell, \bar{z}_\ell)) : \ell \in M\} \cup \{d_\ell \neq \check{m} : \ell, m \in M\},$$

where the  $d_\ell$ 's are new constant symbols. Since  $T \supseteq \text{PA}^-$ , we know from Lemma 3.1 that  $(M, \mathcal{X}) \models \text{Con}(\hat{T})$ . Using Corollary 3.3, find a consistent complete Henkinized theory  $T^* \supseteq \hat{T}$  in  $\mathcal{X}$ . Let  $K$  be the Henkin model of  $T^*$ . Standard arguments show that conditions (1), (2) hold, and  $K \supseteq M$ .

We demonstrate  $K \supseteq_e M$ . Let  $c$  be a Henkin constant in the language of  $T^*$ . It suffices to show

$$(M, \mathcal{X}) \models \forall a (\ulcorner c < \check{a} \urcorner \in T^* \rightarrow \exists m < a \ulcorner c = \check{m} \urcorner \in T^*).$$

We do this by  $\Sigma_0^0$  induction on  $a$ . The statement is trivial when  $a = 0$  because the hypothesis in this case contradicts the axioms of  $\text{PA}^-$ . For the induction step, suppose  $(M, \mathcal{X}) \models \ulcorner c < \check{a} \urcorner \in T^* \rightarrow \exists m < a \ulcorner c = \check{m} \urcorner \in T^*$  and  $\ulcorner c < \check{a} + 1 \urcorner \in T^*$ . Recall that  $\text{PA}^- \vdash \forall z \forall x < z + 1 (x < z \vee x = z)$  — see, for example, Proposition 2.1 in [35]. Our model  $(M, \mathcal{X})$  knows this too. So either  $\ulcorner c < \check{a} \urcorner \in T^*$  or  $\ulcorner c = \check{a} \urcorner \in T^*$ . In the latter case, we are done outright. In the former case, we are done by the induction hypothesis.

For (3), let  $L$  be a definable cut of  $K$ . Find  $\ell \in \mathbb{N}$  and Henkin constants  $\bar{c}$  such that  $L = \{v \in K : K \models \psi_\ell(v, \bar{c}^K)\}$ . On the one hand,

$$(M, \mathcal{X}) \models \ulcorner \psi_\ell(0, \bar{c}) \urcorner \in T^* \wedge \forall m (\ulcorner \psi_\ell(\check{m}, \bar{c}) \urcorner \in T^* \rightarrow \ulcorner \psi_\ell(\check{m} + 1, \bar{c}) \urcorner \in T^*).$$

So  $\Sigma_0^0$  induction in  $(M, \mathcal{X})$  implies  $M = \{m \in M : \ulcorner \psi_\ell(\check{m}, \bar{c}) \urcorner \in T^*\} \subseteq L$ . On the other hand, the definition of  $\hat{T}$  ensures  $d_\ell^K \in L \setminus M$ . Hence  $L \supseteq M$ .

It remains to show recursive saturation. Let  $(\varphi_i(v))_{i \in \mathbb{N}}$  be a recursive enumeration of a type  $p(v)$  over  $K$  in which  $K \models \forall v (\varphi_{i+1}(v) \rightarrow \varphi_i(v))$  for all  $i \in \mathbb{N}$ . The fact that  $p$  is a type tells us

$$\ulcorner \exists v \varphi_i(v) \urcorner \in T^*$$

for every  $i \in \mathbb{N}$ . So by overspill on this  $\Sigma_0^0$  formula, we find a small nonstandard  $\nu \in M$  such that  $\ulcorner \exists v \varphi_\nu(v) \urcorner \in T^*$ . If  $c$  is a Henkin constant making  $\ulcorner \varphi_\nu(c) \urcorner \in T^*$ , then  $\ulcorner \varphi_i(c) \urcorner \in T^*$  for every  $i \in \mathbb{N}$ . So the realization of this  $c$  realizes  $p$  in  $K$ .

Now, suppose  $M = \mathbb{N}$ . Following Wilmers [76, Theorem 2.29], we consider

$$T' = \text{PA} \cup \{\text{Con}(T \cap n) : n \in \mathbb{N}\},$$

where  $T \cap n$  denotes the set of all elements of  $T$  with Gödel numbers less than  $n$ . Notice  $T' \in \mathcal{X}$  by  $\Delta_1^0$  comprehension. A simple modification of the arguments above (as in the proof of Proposition 4.4) gives us  $M' \models T'$  with  $\text{ElemDiag}(M') \in \mathcal{X}$  and  $T \in \text{SSy}(M')$ . With a fixed code for  $T$  in mind, one can extend the notation  $T \cap n$  to all nonstandard  $n \in M'$ . Apply  $\Pi_1$  overspill to find  $\nu \in M' \setminus \mathbb{N}$  such that  $M' \models \text{Con}(T \cap \nu)$ . The required extension  $K$  can then be obtained by repeating again the arguments in the previous paragraphs with the *nonstandard* model  $(M', \text{Def}(M')) \models \text{WKL}_0^*$  and the theory  $T \cap \nu \supseteq \text{PA}^-$ . The fact that the second-order part of this nonstandard model consists entirely of definable subsets of  $M'$  helps in establishing (2).  $\square$

The usual notion of consistency is too strong for many of our arguments. We thus adopt a cut-free deduction system, which is weaker than the usual one without the totality of superexponentiation [47, 66]. With  $\text{exp}$  at hand, it does not matter which cut-free system we choose. Following Wilkie and Paris [73], we employ the *tableau deduction system*. For the details of this system and its arithmetization, see the book by Bell and Machover [12] and the Master's Thesis of Kalsbeek's [34], for example. By  $\text{TabCon}(T)$ , we denote a suitable  $\Pi_1^0$  formula which expresses within  $\text{RCA}_0^*$  the tableau consistency of  $T$ . Let  $\text{TabProv}(T, \theta)$  be an abbreviation for  $\neg \text{TabCon}(T + \neg\theta)$ . Both of these make sense in  $\text{I}\Delta_0 + \text{exp}$  when  $T$  is definable. The *cut rule* in this context allows one to deduce in one extra step the inconsistency of a theory  $T$  from the inconsistencies of  $T + \theta$  and  $T + \neg\theta$ , whenever  $\theta$  is a formula. In this situation, the formula  $\theta$  is called the *cut formula*. Unless explicitly stated, the use of the cut rule is not permitted in tableau deductions.

*Superexponentiation* is defined by  $2_0 = 1$  and  $2_{x+1} = 2^{2^x}$ . We denote by  $\text{supexp}$  a fixed  $\Pi_2$  sentence that expresses the totality of superexponentiation over  $\text{I}\Delta_0$ . Since cut elimination makes deductions (at most) superexponentially larger [67], tableau provability coincides with usual provability over  $\text{RCA}_0^* + \text{supexp}$ . This immediately gives tableau versions of Lemma 3.1, Lemma 3.2, Corollary 3.3 and Proposition 3.4 if  $\text{supexp}$  is added to  $\text{WKL}_0^*$  there. Without  $\text{supexp}$ , we (may have to) settle for weaker conclusions.

The syntactical version of the Compactness Theorem is not affected at all, as indicated in the following lemma. The proof is nearly identical.

**Lemma 3.1'.**  $\text{RCA}_0^*$  proves that if every bounded coded subset of a theory  $T$  is tableau consistent, then  $T$  is tableau consistent.  $\square$

We run into a problem when we try to show Lindenbaum's Lemma for the tableau system in  $\text{WKL}_0^*$ . In the proof of Lemma 3.2, the cut rule was used to show the claim that the tree there has a node at every level. In the presence of  $\text{supexp}$ , all these applications of the cut rule can be eliminated. When neither  $\text{supexp}$  nor the cut rule is available, it is not clear how one can show this claim.

Fortunately, we can still deal with formulas with *standard shapes*; these are what matter for most applications of the Arithmetized Completeness Theorem. Define the *cut rank* of a deduction to be the maximum number of logical symbols in a cut formula plus one, or zero if the deduction is cut-free. (By *logical symbols*, we mean sentential connectives and quantifiers.) Given any deduction of non-zero cut rank, one can construct another one of strictly smaller cut rank with the same hypotheses



and conclusions, at the expense of an exponential growth in size [67]. Therefore, the theory  $\text{I}\Delta_0 + \text{exp}$  can afford to eliminate the cuts in all deductions of *standard* cut ranks. For the details, consult Section 7.1 in Visser [70].

**Lemma 3.2'.** For every  $n \in \mathbb{N}$ , the theory  $\text{WKL}_0^*$  proves that every tableau consistent theory has a tableau consistent extension which is complete for all formulas with at most  $n$  logical symbols.

*Proof.* Proceed as in the proof of Lemma 3.2, but instead of dealing with all formulas, restrict ourselves to formulas with at most  $n$  logical symbols.  $\square$

By an iterated application of Lemma 3.2', one can complete, in the sense of the real world, any tableau consistent theory in a model of  $\text{WKL}_0^*$ . The resulting completion may not be entirely internal to the model, as can be seen in the corollary below. On the plus side, this construction allows one to realize countably many types in the ground model simultaneously.

**Corollary 3.3'.** Fix  $(M, \mathcal{X}) \models \text{WKL}_0^*$  and a theory  $T \in \mathcal{X}$  such that  $(M, \mathcal{X}) \models \text{TabCon}(T)$ . Let  $\mathcal{L}$  be the language of  $T$  and  $P_0(v), P_1(v), P_2(v), \dots \in \mathcal{X}$  of which all elements are  $\mathcal{L}$  formulas in the sense of  $(M, \mathcal{X})$  with a single free variable  $v$ . Then there exist  $T^* \supseteq T$  and a language  $\mathcal{H} \supseteq \mathcal{L}$  with the following properties.

- (1) Every element of  $T^*$  is an  $\mathcal{H}$  formula.
- (2) Whenever  $n \in \mathbb{N}$  and  $c_1, c_2, \dots, c_n$  are constant symbols in  $\mathcal{H}$ ,

$\{\theta(c_1, c_2, \dots, c_n) \in T^* : \theta \text{ is an } \mathcal{L} \text{ formula in the sense of } (M, \mathcal{X}) \text{ with at most } n \text{ logical symbols}\}$

is in  $\mathcal{X}$ , and is known to be tableau consistent to  $(M, \mathcal{X})$ .

- (3)  $T^*$  decides every  $\mathcal{H}$  formula with a standard number of logical symbols.
- (4) For every  $\mathcal{H}$  formula  $\psi(v)$  with a standard number of logical symbols, there is a constant symbol  $c$  such that  $T^*$  contains  $\exists v \psi(v) \rightarrow \psi(c)$ .
- (5) For every  $n \in \mathbb{N}$ , either  $T^* \supseteq P_n(c)$  for some constant symbol  $c$ , or  $(M, \mathcal{X}) \models \text{TabCon}(S) \wedge \neg \text{TabCon}(S \cup P_n(d))$  for some  $S \subseteq T^*$  in  $\mathcal{X}$  and some fresh constant symbol  $d$ .

*Proof.* Find theories  $T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$  in  $\mathcal{X}$  such that for every  $n \in \mathbb{N}$ ,

- $(M, \mathcal{X}) \models \text{TabCon}(T_n)$ ;
- $T_{n+1}$  is complete for formulas with at most  $n$  logical symbols;
- for every formula  $\psi(v)$  in the language of  $T_n$  with at most  $n$  logical symbols, there is a constant symbol  $c$  such that  $T_{n+1}$  contains  $\exists v \psi(v) \rightarrow \psi(c)$ ; and
- either  $T_{n+1} \supseteq P_n(c)$  for some constant symbol  $c$ , or  $(M, \mathcal{X}) \models \neg \text{TabCon}(T_n \cup P_n(d))$  for any fresh constant symbol  $d$ .

This can be achieved by an iterative construction, at each stage of which we include the next  $P_n$  if possible, add the appropriate Henkin axioms, and then apply Lemma 3.2'. Here we employ the argument in the proof of Corollary 3.3 to show that the addition of these Henkin axioms preserves tableau consistency. The use of the cut rule there can be eliminated because all the cut formulas involved have at most  $n + 1$  logical symbols. Now  $T^* = \bigcup_{n \in \mathbb{N}} T_n$  is what we want, because if  $\bar{c}$  are constant symbols in the language of  $T_{n+1}$ , then

$\{\theta(\bar{c}) \in T^* : \theta \text{ is an } \mathcal{L} \text{ formula in the sense of } (M, \mathcal{X}) \text{ with at most } n \text{ logical symbols}\}$   
 $= \{\theta(\bar{c}) \in T_{n+1} : \theta \text{ is an } \mathcal{L} \text{ formula in the sense of } (M, \mathcal{X}) \text{ with at most } n \text{ logical symbols}\} \in \mathcal{X}$

by  $\Delta_1^0$  comprehension.  $\square$

*Remark 3.5.* Since the extension  $T^*$  given by Corollary 3.3' may not be in  $\mathcal{X}$  outright, it seems difficult to achieve full recursive saturation as in Proposition 3.4

without any additional assumption on  $(M, \mathcal{X})$  or  $T$ . The same also applies to partial notions of recursive saturation, such as  $\Delta_0$  recursive saturation, in which formulas may involve an unbounded number of quantifier alterations; cf. Gerhardy [29, Corollary 12]. If one ignores condition (5), then a sufficient condition on  $(M, \mathcal{X})$  for making the extension  $T^*$  in Corollary 3.3' in  $\mathcal{X}$  is short  $\Pi_2$  recursive saturation, because this gives us a nonstandard  $\nu \in M$  such that the statement

for every deduction of cut rank at most  $\nu$ , there exists a cut-free deduction which has the same hypotheses and conclusions

is true in  $M$ . Whether one can weaken short  $\Pi_2$  recursive saturation to some amount of recursive saturation possessed by all nonstandard models of  $\text{I}\Sigma_1$  is not clear. In the absence of extra assumptions, what remains is the undefinability of the standard cut. See also the paragraph after Corollary 4.3.

**Proposition 3.4'.** Fix  $(M, \mathcal{X}) \models \text{WKL}_0^*$  and a tableau consistent theory  $T \supseteq \text{PA}^-$  in  $(M, \mathcal{X})$ . Let  $p_0(v), p_1(v), p_2(v), \dots$  be sets of  $\mathcal{L}_A$  formulas with parameters from  $M$  such that for every  $n \in \mathbb{N}$ , the second-order part  $\mathcal{X}$  contains the set  $P_n(v)$ , obtained from  $p_n(v)$  by replacing each occurrence of a parameter  $m \in M$  with the corresponding canonical term  $\check{m}$ , and

$$(M, \mathcal{X}) \models \forall T' \supseteq T \ (\text{TabCon}(T') \rightarrow \exists d \text{TabCon}(T' \cup P_n(d))).$$

Then there is a proper end extension  $K \supseteq_e M$  with the following properties.

- (1) If  $\theta(\check{m}_1, \check{m}_2, \dots, \check{m}_k) \in T$  where  $\theta(\bar{x})$  has a standard shape, then  $K \models \theta(m_1, m_2, \dots, m_k)$ .
- (2) For each  $n \in \mathbb{N}$  and each  $\bar{c} \in K$ , there exists  $S \in \mathcal{X}$  such that

$$\{(j, s) \in S : j \in \mathbb{N}\} = \{(\ulcorner \theta^\ulcorner, [\bar{m}] \urcorner) \in M : \theta \in \mathcal{L}_A \text{ with at most } n \text{ logical symbols and } K \models \theta(\bar{m}, \bar{c})\}.$$

- (3) All definable cuts of  $K$  include  $M$  properly.
- (4) The type  $p_n(v)$  is realized in  $K$  for every  $n \in \mathbb{N}$ .

*Proof.* Proceed as in the proof of Proposition 3.4, using Corollary 3.3' in place of Corollary 3.3. We no longer need to deal with the case when  $M = \mathbb{N}$  separately because we do not need to ensure recursive saturation here. When demonstrating  $K \supseteq_e M$ , notice we have

$$\{m \in M : \ulcorner c < \check{m} \urcorner \in T^*\} \quad \text{and} \quad \{m \in M : \ulcorner c = \check{m} \urcorner \in T^*\}$$

in  $\mathcal{X}$  from Corollary 3.3'(2), so that we can stay within  $(M, \mathcal{X})$  in the induction. Similarly, since

$$\{m \in M : \ulcorner \psi_\ell(\check{m}, \bar{c}) \urcorner \in T^*\}$$

is in  $\mathcal{X}$ , all definable cuts of  $K$  can be shown to include  $M$ . □

Proposition 3.4' gives a proof of the  $\Sigma_1$  completeness of  $\text{PA}^-$  within  $\text{I}\Delta_0 + \text{exp}$ . This previously known fact appears as Lemma V.5.24(i) in [32]. The question whether one can weaken  $\text{I}\Delta_0 + \text{exp}$  to  $\text{I}\Delta_0 + \Omega_1$  here is related to the NP-coNP problem; see Visser [71] and Verbrugge–Visser [69].

**Corollary 3.6** (folklore).  $\text{I}\Delta_0 + \text{exp} \vdash \forall z (\theta(z) \rightarrow \text{TabProv}(\text{PA}^-, \theta(\check{z})))$  for every  $\theta \in \Sigma_1$ .

*Proof.* Notice the sentence we want to deduce from  $\text{I}\Delta_0 + \text{exp}$  is  $\Pi_2$ . So by Proposition 2.4 and Corollary 2.8, it suffices to work inside a model  $(M, \mathcal{X}) \models \text{WKL}_0^*$ . Suppose  $(M, \mathcal{X}) \not\models \text{TabProv}(\text{PA}^-, \theta(\check{z}))$ . By the definition of tableau provability, this means  $(M, \mathcal{X}) \models \text{TabCon}(\text{PA}^- + \neg\theta(\check{z}))$ . Using Proposition 3.4', find an end extension  $K \supseteq_e M$  that satisfies  $\text{PA}^- + \neg\theta(z)$ . As  $\neg\theta \in \Pi_1$ , we conclude  $M \models \neg\theta(z)$ . □

The main advantage of cut-free deductions over usual deductions is that they have the *subformula property*. In the case of the tableau deduction system, this means all formulas appearing in a deduction are subformulas of the *assumptions*, i.e., those formulas whose appearances are not justified by any deduction rule. Assumptions are called *initial formulas* in Bell–Machover [12].

The next fact explains why the usual deduction system is too strong for us. The first part can be viewed as a uniform  $\Pi_2$  reflection scheme. Since  $\text{exp} \in \Pi_2$ , one can use Theorem 4.27 in the next section together with Corollary 2.8 and Proposition 3.4' to show that the analogous  $\Sigma_2$  reflection scheme is not provable in  $\text{I}\Delta_0 + \text{exp}$ ; cf. Adamowicz [5] and Willard [75].

**Theorem 3.7** (Wilkie–Paris, Visser). For every formula  $\varphi \in \Sigma_2$  and every  $k \in \mathbb{N}$ ,

$$\text{I}\Delta_0 + \text{exp} \vdash \forall z (\varphi(z) \rightarrow \text{TabCon}(\text{I}\Delta_0 + \Omega_k + \varphi(\check{z}))),$$

but  $\text{I}\Delta_0 + \text{exp} \not\vdash \text{Con}(Q)$ .

*Proof.* The former is Theorem 7.2.3 of Visser [70], the proof of which is a slight variant of that of Lemma 8.10 in Wilkie–Paris [73]. The latter is a part of Corollary 8.14 in [73].  $\square$

*Remark 3.8.* Recall Harvey Friedman, and independently, Paris [48] showed that  $\text{B}\Sigma_{n+1}$  is  $\Pi_{n+2}$ -conservative over  $\text{I}\Sigma_n$  for every  $n \in \mathbb{N}$ . Clote, Hájek and Paris [19, page 216] asked whether this conservation result can be proved in  $\text{I}\Delta_0 + \text{exp}$ , and they [19, Corollary 17] proved that

$$\text{I}\Delta_0 + \Omega_1 \vdash \forall \sigma \in \Pi_{n+2} (\text{TabProv}(\text{B}\Sigma_{n+1}, \sigma) \rightarrow \text{Prov}(\text{I}\Sigma_n, \sigma))$$

whenever  $n \geq 1$ ; cf. Question 36 in [18]. As far as we know, their question is still open. Hájek's interpretation [31, Theorem 2.6] of  $\text{B}\Sigma_{n+1}$  in  $\text{I}\Sigma_n$  shows if  $n \geq 1$ , then

$$\text{I}\Delta_0 + \Omega_1 \vdash \forall \sigma \in \Sigma_{n+1} (\text{Prov}(\text{B}\Sigma_{n+1}, \sigma) \rightarrow \text{Prov}(\text{I}\Sigma_n, \sigma)).$$

The same is true for the tableau system if  $\text{I}\Delta_0 + \Omega_1$  is strengthened to  $\text{I}\Delta_0 + \text{exp}$ . If  $\text{B}\Sigma_1 + \Omega_k$  is finitely axiomatizable for some  $k \in \mathbb{N}$ , then a similar argument involving Solovay's method of shortening cuts [32, Theorem III.3.5 and Theorem V.5.7] shows

$$\text{I}\Delta_0 + \text{exp} \vdash \forall z (\text{TabCon}(\text{I}\Delta_0 + \varphi(\check{z})) \rightarrow \text{TabCon}(\text{B}\Sigma_1 + \Omega_k + \varphi(\check{z})))$$

for all  $\varphi \in \Pi_1$ , and so one can replace  $\text{I}\Delta_0 + \Omega_k$  in Theorem 3.7 by  $\text{B}\Sigma_1 + \Omega_k$ . However, as pointed out by the anonymous referee, the question whether  $\text{B}\Sigma_1 + \Omega_k$  can be finitely axiomatized for some  $k \in \mathbb{N}$  is open; cf. Question A in [18].

#### 4. VARIETIES OF END EXTENSIONS

In this section, we study and construct end extensions of models of  $\text{B}\Sigma_1 + \text{exp}$ . One of our foremost interests is in determining what theories, particularly how much of PA, can be arranged to be satisfied in an end extension. In some cases, we can actually tell whether the end extension can be expanded to a model of a certain theory; see Theorem 4.20. We also consider a wide range of other requirements on the end extension, for example, how saturated it is, how elementary it is, whether it is  $\Sigma_1$ -closed, whether it is isomorphic to the ground model, whether it can use a particular cut to code the ground model, and which sets are coded. We try to keep to a minimum the use of extra model theoretic assumptions on the ground model such as  $\text{WKL}_0^*$ -expandability, countability, and recursive saturation. However, it will be shown that such extra assumptions are necessary on a number of occasions.

We begin with a slight strengthening of Theorem 2.7.

**Proposition 4.1.** Let  $M$  be a  $\text{WKL}_0^*$ -expandable model of  $\text{B}\Sigma_1 + \text{exp}$ . Then for every  $k \in \mathbb{N}$ , there exists  $K \models \text{B}\Sigma_1 + \Omega_k$  properly end extending  $M$  in which  $\mathbb{N}$  is not definable.

*Proof.* Apply Proposition 3.4' to  $\text{I}\Delta_0$  within some  $(M, \mathcal{X}) \models \text{WKL}_0^*$  to obtain  $K_0 \supseteq_e M$  satisfying the conditions there. If  $K_0 \models \text{B}\Sigma_1 + \Omega_k$ , then we are already done. So suppose not. Then  $K_0$  has a proper definable cut, say  $K'_0$ . Now use Solovay's method [32, Theorem III.3.5] to shorten  $K'_0$  to a definable cut  $K$  closed under  $+$ ,  $\times$  and  $\omega_k$ . These make  $K \models \text{B}\Sigma_1 + \Omega_k$ . Condition (3) in Proposition 3.4' guarantees that  $K$  must be a proper extension of  $M$ . Since  $\mathbb{N}$  is not definable in  $K_0$ , it remains so in  $K$  because  $K$  is definable in  $K_0$ .  $\square$

The corollary below reveals a tight link between the work by the Paris–Wilkie school in first-order arithmetic and that by the Friedman–Simpson school in second-order arithmetic, since, for example, it shows that the case  $n = 0$  of Theorem 2.11 about the expandability of countable models of  $\text{B}\Sigma_1 + \text{exp}$  to models of  $\text{WKL}_0^*$ , due originally to Simpson and Smith [60], can be derived without the use of forcing from a theorem in the Wilkie–Paris paper [74] asserting that every countable model of  $\text{B}\Sigma_1 + \text{exp}$  has a proper end extension to a model of  $\text{I}\Delta_0$ . As pointed out in Corollary 2.8, all models of  $\text{B}\Sigma_1 + \text{exp}$  of countable cofinality can be shown to be  $\text{WKL}_0^*$ -expandable by the same argument.

**Corollary 4.2.** A model of  $\text{B}\Sigma_1 + \text{exp}$  is  $\text{WKL}_0^*$ -expandable if and only if it has a proper end extension satisfying  $\text{I}\Delta_0$ .

*Proof.* Combine Proposition 2.6 and Proposition 4.1.  $\square$

It is not hard to manufacture recursively saturated end extensions from Proposition 4.1 when the models are countable.

**Corollary 4.3.** Let  $M$  be a countable model of  $\text{B}\Sigma_1 + \text{exp}$ . Then for every  $k \in \mathbb{N}$ , there exists a recursively saturated  $K \models \text{B}\Sigma_1 + \Omega_k$  properly end extending  $M$ .

*Proof.* This follows from Corollary 2.8, Proposition 4.1 and Corollary 2.10.  $\square$

On the one hand, as indicated in the remark before Proposition 3.4', there are difficulties in making the end extension  $K$  in Proposition 4.1 recursively saturated in the general case when  $M$  may be uncountable. To get some recursive saturation in this case, it suffices to have, for some  $k, n \in \mathbb{N}$ , a definable satisfaction predicate for  $\Sigma_n$  formulas in all models of  $\text{B}\Sigma_1 + \Omega_k$ . In fact, such a satisfaction predicate enables one to show the equivalence between  $\Sigma_n$  recursive saturation for every  $n \in \mathbb{N}$  and the undefinability of  $\mathbb{N}$  in every nonstandard model of  $\text{B}\Sigma_1 + \Omega_k$ . It should be noted, however, that this additional hypothesis is related to open questions in complexity theory [49, 6]. The whole problem disappears if  $M$  here satisfies  $\text{Con}(\text{I}\Delta_0)$ .

On the other hand, if  $\text{I}\Delta_0 + \Omega_{k+1}$  is  $\Pi_1$ -conservative over  $\text{I}\Delta_0 + \Omega_k$  for every  $k \in \mathbb{N}$  provably in  $\text{I}\Delta_0 + \text{exp}$ , and if one is willing to sacrifice the saturation condition in Proposition 4.1, then we can require the end extension to satisfy  $\Omega_\omega = \{\Omega_k : k \in \mathbb{N}\}$ . The question whether any of these conservation results holds (in the real world) is again open [4]. Another way to make  $K \models \Omega_\omega$  is to assume, in addition, that the ground model  $M$  is short  $\Pi_1$ -recursively saturated; see the end of this section for a related discussion.

A theorem from Scott and Tennenbaum [57] states that there is no minimal PA degree. Since the theory of PA degrees is closely related to that of models of  $\text{WKL}_0$  (as expounded in §2 of Marker's thesis [44] or Chapter 3 of Hirschfeldt's book [33], for example), one can reformulate the Scott–Tennenbaum theorem in terms of models of  $\text{WKL}_0^*$ . Simpson [59, Corollary VIII.2.7] showed that this reformulation can be proved within  $\text{WKL}_0$ . (See Simpson [58] for some further results

on this line.) Using Proposition 3.4, we weaken  $\text{WKL}_0$  almost to  $\text{WKL}_0^*$ . Recall that a *coded  $\omega$ -submodel* of a model  $(M, \mathcal{X}) \models \text{RCA}_0^*$  is a structure in the language of second-order arithmetic of the form

$$(M, \{(Y)_c : c \in M\}),$$

where  $(Y)_c$  denotes the  $c$ th column of  $Y \in \mathcal{X}$ .

**Proposition 4.4.** Let  $(M, \mathcal{X}) \models \text{WKL}_0^* + \text{Con}(\text{I}\Delta_0)$ . Then for every  $D \in \mathcal{X}$ , there exists a coded  $\omega$ -submodel of  $(M, \mathcal{X})$  that satisfies  $\text{WKL}_0^*$  and contains  $D$ .

*Proof.* Using Lemma 3.2, find a consistent  $\mathcal{L}_A$  theory  $T_0$  in  $(M, \mathcal{X})$  which extends  $\text{I}\Delta_0$  and decides  $\check{m} \in_{\text{Ack}} \check{s}$  for all  $m, s \in M$ . By  $\Delta_1^0$  comprehension in  $(M, \mathcal{X})$ , the theory

$$T = T_0 \cup \{\check{m} \in_{\text{Ack}} d : m \in D\} \cup \{\check{m} \notin_{\text{Ack}} d : m \in M \setminus D\},$$

where  $d$  is a new constant symbol, is in  $\mathcal{X}$ . We show  $(M, \mathcal{X}) \models \text{Con}(T)$ . In view of Lemma 3.1, it suffices to show that every bounded coded subset of  $T$  is consistent in  $(M, \mathcal{X})$ .

Let  $b \in M$  be arbitrary. Using  $\text{I}\Sigma_0^0 + \text{exp}$  as in Proposition 2.1, find  $s \in M$  such that

$$(M, \mathcal{X}) \models \forall m < b (m \in_{\text{Ack}} s \leftrightarrow m \in D).$$

By Corollary 3.6, we know  $\ulcorner \check{m} \in_{\text{Ack}} \check{s} \urcorner \in T_0$  whenever  $m \in D$  below  $b$ , and  $\ulcorner \check{m} \notin_{\text{Ack}} \check{s} \urcorner \in T_0$  whenever  $m \in M \setminus D$  below  $b$ . As a result, the theory

$$\begin{aligned} & T_0 \cup \{d = \check{s}\} \\ & \cup \{\check{m} \in_{\text{Ack}} d : m \in D \wedge m < b\} \\ & \cup \{\check{m} \notin_{\text{Ack}} d : m \in M \setminus D \wedge m < b\} \end{aligned}$$

is essentially the same as  $T_0$ , and so must be consistent in  $(M, \mathcal{X})$  too.

Apply Proposition 3.4 to our  $T$  to get an end extension  $K \supseteq_e M$ . Let  $e$  be a surjection  $M \rightarrow K$  given by condition (2) in Proposition 3.4, so that the set

$$Y = \{(m, c) \in M : K \models m \in_{\text{Ack}} e(c)\}$$

is in  $\mathcal{X}$ . If  $\check{d}$  is a preimage of the realization of  $d$  under  $e$ , then

$$D = (Y)_{\check{d}} \in \{(Y)_c : c \in M\} = \text{SSy}_M(K).$$

Hence we are done by Proposition 2.6.  $\square$

As the anonymous referee suggests, it is worth pointing out the following corollary.

**Corollary 4.5.** Every  $(M, \mathcal{X}) \models \text{WKL}_0^* + \text{Con}(\text{I}\Delta_0)$  has a countably generated submodel  $(M, \mathcal{X}_0) \models \text{WKL}_0^*$ .

*Proof.* Set  $D_0 = \emptyset$ . Given  $D_n \in \mathcal{X}$ , apply Proposition 4.4 to find  $D_{n+1} \in \mathcal{X}$  which codes an  $\omega$ -submodel of  $(M, \mathcal{X})$  that satisfies  $\text{WKL}_0^*$  and contains  $D_n$ . Then  $\{D_n : n \in \mathbb{N}\}$  clearly  $\Delta_1^0$ -generates a model  $(M, \mathcal{X}_0) \models \text{WKL}_0^*$  with  $\mathcal{X}_0 \subseteq \mathcal{X}$ .  $\square$

The next theorem shows how one can adapt the proof of Proposition 4.4 to demonstrate that  $\text{WKL}_0^*$  is precisely the second-order theory of proper cuts closed under exponentiation in models of  $\text{I}\Delta_0$ , as claimed by Ratajczyk [52, Corollary 4]. The proof is in essence similar to that of Theorem 3.2 in Enayat [26]; cf. the proof of Theorem 2.8 in Smoryński [64] and the proof of Theorem 13.6 in Kaye [35]. The special case for  $\omega$ -standard models was first shown by Scott [56] without the saturation condition.

**Theorem 4.6.** For every countably generated  $(M, \mathcal{X}) \models \text{WKL}_0^*$  and every  $k \in \mathbb{N}$ , there exists  $K \models \text{B}\Sigma_1 + \Omega_k$  properly end extending  $M$  such that  $\text{SSy}_M(K) = \mathcal{X}$  and  $\mathbb{N}$  is not definable in  $K$ .

*Proof.* Let  $(D_n)_{n \in \mathbb{N}}$  enumerate a subset of  $\mathcal{X}$  that  $\Delta_1^0$ -generates  $(M, \mathcal{X})$ . Apply Proposition 3.4' to  $\text{I}\Delta_0$  to obtain an end extension  $K \supseteq_e M$  satisfying the conditions there, where

$$p_n(v) = \{m \in_{\text{Ack}} v : m \in D_n\} \cup \{m \notin_{\text{Ack}} v : m \in M \setminus D_n\}$$

for all  $n \in \mathbb{N}$ . As in the proof of Proposition 4.4, all these  $p_n$ 's can be shown to satisfy the hypotheses in the statement of Proposition 3.4'. The fact that  $\mathcal{X} \supseteq \text{SSy}_M(K)$  follows from condition (2) there and  $\Delta_1^0$  comprehension. Finally, the proof of Proposition 4.1 shows how one can shorten  $K$  to a model of  $\text{B}\Sigma_1 + \Omega_k$  properly end extending  $M$  while preserving the non-definability of  $\mathbb{N}$ .  $\square$

*Remark 4.7.* As announced by the first author at the end of a previous paper [24], the countable models of  $\text{WKL}_0^*$  are precisely those models of the form  $(M, \text{SSy}_M(K))$  where  $K$  is a countable model of  $\text{I}\Delta_0$  and  $M$  is the greatest initial segment pointwise fixed by a nontrivial automorphism of  $K$ . Theorem 4.6 provides the last ingredient for the proof of this characterization.

Let us present two applications of Theorem 4.6. They are reminiscent of Kirby's analysis of semiregular cuts [36, Chapter 2]; see also Remark 1 in Paris–Kirby [50, page 208].

**Definition.** Let  $I \subseteq_e M \subseteq_e K \models \text{I}\Delta_0$ . We say that  $M$  is *I-coded in K* if

$$M = \sup\{f(i) \in K : i \in I\}$$

for some function  $f: I \rightarrow K$  coded in  $K$ . Dually, we say that  $M$  is *I\*-coded in K* if

$$M = \inf\{f(i) \in K : i \in I\}$$

for some function  $f: I \rightarrow K$  coded in  $K$ .

**Corollary 4.8.** For a countably generated  $(M, \mathcal{X}) \models \text{WKL}_0^*$  and  $I \subsetneq_e M$ , the following are equivalent.

- (a)  $I$  is  $\Sigma_1^0$ -definable in  $(M, \mathcal{X})$ .
- (b) In every  $K \models \text{I}\Delta_0$  properly end extending  $M$  where  $\text{SSy}_M(K) = \mathcal{X}$ , the set  $M$  is  $I$ -coded.
- (c) In some  $K \models \text{I}\Delta_0$  properly end extending  $M$  where  $\text{SSy}_M(K) = \mathcal{X}$ , the set  $M$  is  $I$ -coded.

*Proof.* First, consider (a)  $\Rightarrow$  (b). Let  $\varphi(i, v) \in \Sigma_0^0$ , possibly with parameters from  $(M, \mathcal{X})$ , such that

$$I = \{i \in M : (M, \mathcal{X}) \models \exists v \varphi(i, v)\}.$$

Take any  $K \models \text{I}\Delta_0$  properly end extending  $M$  in which  $\text{SSy}_M(K) = \mathcal{X}$ . Replacing the set parameters in  $\varphi$  by their codes, obtain  $\psi(i, v) \in \Delta_0$  with parameters from  $K$  such that for all  $i, v \in M$ ,

$$(M, \mathcal{X}) \models \varphi(i, v) \Leftrightarrow K \models \psi(i, v).$$

Fix  $a \in M \setminus I$ . Let  $d \in K \setminus M$  small enough such that  $2^{(a, d)}$  exists in  $K$ . For each  $i < a$ , define

$$f(i) = \begin{cases} (\min v \leq d)(K \models \psi(i, v)), & \text{if it exists;} \\ d, & \text{otherwise.} \end{cases}$$

This function  $f$  is coded in  $K$  by Proposition 2.1. By our choices of  $\varphi$  and  $\psi$ , we know  $f(i) \in M$  for every  $i \in I$ . Let  $b \in M$  large enough such that  $(M, \mathcal{X}) \models \exists i < a \exists v < b \varphi(i, v)$ . Then

$$j = (\max i < a)((M, \mathcal{X}) \models \exists v < b \varphi(i, v))$$

exists by  $\text{I}\Sigma_0^0$  in  $(M, \mathcal{X})$ . It must be in  $I$  in view of the choice of  $\varphi$ . Thus  $b < f(j+1) \in M$  by the maximality of  $j$ . As  $b$  can be arbitrarily large in  $M$ , we conclude  $M = \sup\{f(i) \in K : i \in I\}$ , as required.

The implication (b)  $\Rightarrow$  (c) follows directly from Theorem 4.6. This is the only part that requires  $(M, \mathcal{X})$  to be countably generated.

Finally, we show (c)  $\Rightarrow$  (a). Let  $a \in M \setminus I$  and  $f: a \rightarrow K$  coded in  $K$  such that  $M = \sup\{f(i) \in K : i \in I\}$ . Without loss of generality, assume  $f$  is non-decreasing. If  $F = \{(i, f(i)) \in M : i < a\}$ , then

$$I = \{i < a : f(i) \in M\} = \{i < a : (M, \mathcal{X}) \models \exists v \langle i, v \rangle \in F\}$$

and  $F \in \text{SSy}_M(K) = \mathcal{X}$ . So  $I$  is  $\Sigma_1^0$ -definable in  $(M, \mathcal{X})$ .  $\square$

**Corollary 4.9.** For a countably generated  $(M, \mathcal{X}) \models \text{WKL}_0^*$  and  $I \subsetneq_e M$ , the following are equivalent.

- (a)  $I$  is  $\Pi_1^0$ -definable in  $(M, \mathcal{X})$ .
- (b) In every  $K \models \text{I}\Delta_0$  properly end extending  $M$  where  $\text{SSy}_M(K) = \mathcal{X}$ , the set  $M$  is  $I^*$ -coded.
- (c) In some  $K \models \text{I}\Delta_0$  properly end extending  $M$  where  $\text{SSy}_M(K) = \mathcal{X}$ , the set  $M$  is  $I^*$ -coded.

*Proof.* This is very similar to the previous one. So we content ourselves with merely a sketch of the proof for (a)  $\Rightarrow$  (b). Let  $\varphi(i, v) \in \Sigma_0^0$ , possibly with parameters from  $(M, \mathcal{X})$ , such that

$$I = \{i \in M : (M, \mathcal{X}) \models \forall v \varphi(i, v)\}.$$

Follow the previous proof, but define  $f: a \rightarrow K$  by

$$f(i) = \begin{cases} (\max b \leq d)(K \models \forall v < b \psi(i, v)), & \text{if it exists;} \\ d, & \text{otherwise.} \end{cases}$$

Let  $b_0 \in K \setminus M$  small enough such that  $K \models \exists i < a \forall v < b_0 \psi(i, v)$ . Then

$$j = (\max i < a)(K \models \forall v < b_0 \psi(i, v))$$

exists by  $\text{I}\Delta_0$  in  $K$ . It must be in  $I$  because of our choice of  $\varphi$  and  $\psi$ . So  $b_0 > f(j+1) > M$  by the maximality of  $j$ . Since  $b_0$  can be arbitrarily small in  $K \setminus M$ , we conclude  $M = \inf\{f(i) \in K : i \in I\}$ .  $\square$

As hinted at earlier, the non-definability of the standard cut can be formulated in terms of recursive saturation. For example, it is not hard to see that a model  $M \models \text{I}\Delta_0 + \text{exp}$  is  $\Sigma_{n+1}$ -recursively saturated, where  $n \in \mathbb{N}$ , if and only if  $\mathbb{N}$  is not  $\Sigma_{n+1}$ -definable in  $M$ . Hence, Corollary 4.8 gives a characterization of countably generated models of  $\text{WKL}_0^*$  which are  $\Sigma_1^0$ -recursively saturated. Dually, Corollary 4.9 gives a characterization of those which are short  $\Pi_1^0$ -recursively saturated.

**Definition.** If  $\Gamma$  is a class of formulas, then a *short  $\Gamma$ -recursively saturated* model is one that realizes all recursive types of the form

$$p(v) = \{\varphi_i(v) : i \in \mathbb{N}\} \cup \{v < b\},$$

where the  $\varphi_i$ 's are in  $\Gamma$  and  $b$  is a parameter from the model.

It is again not hard to see that, given any  $n \in \mathbb{N}$ , a nonstandard model  $M \models \text{B}\Sigma_{n+1} + \text{exp}$  is short  $\Pi_{n+1}$ -recursively saturated if and only if  $\mathbb{N}$  is not  $\Pi_{n+1}$ -definable in  $M$ . So every nonstandard model of  $\text{I}\Sigma_{n+1}$  is both  $\Sigma_{n+1}$ - and short  $\Pi_{n+1}$ -recursively saturated. Similarly, in view of Theorem 2.5, every nonstandard model of  $\text{B}\Sigma_{n+1} + \text{exp}$  is either  $\Sigma_{n+1}$ - or short  $\Pi_{n+1}$ -recursively saturated. Notice, however, that plain  $\Pi_n$  recursive saturation is equivalent to  $\Sigma_{n+1}$  recursive saturation for all models of  $\text{I}\Delta_0$ , while short  $\Sigma_{n+1}$  recursive saturation is equivalent to plain  $\Sigma_{n+1}$  recursive saturation for all nonstandard models of  $\text{I}\Delta_0 + \text{exp}$ . One can show using chronic resplendency or the natural interpretation of the structure  $b^2$  in the structure  $b$  that a model  $M \models \text{I}\Delta_0$  is short  $\Delta_0$ -recursively saturated if and only if every  $b \in M$  is recursively saturated as a substructure of  $M$ . As a consequence, every model of  $\text{I}\Delta_0 + \text{exp}$  is short  $\Delta_0$ -recursively saturated by Corollary 2.3. In Section 5, one finds another notion of recursive saturation that corresponds to the  $\Delta_0^-(\Sigma_1)$  definability of  $\mathbb{N}$ .

As commented after Corollary 4.3, some recursive saturation can help find end extensions satisfying a stronger theory. The argument is the same as that in Paris [48, Observations on page 252].

**Corollary 4.10.** Let  $M \models \text{B}\Sigma_1 + \text{exp}$  and  $K \models \text{I}\Delta_0$  properly end extending  $M$  such that  $(M, \text{SSy}_M(K))$  is short  $\Pi_1^0$ -recursively saturated. Then there exists  $K'$  such that  $M \subsetneq_e K' \subsetneq_e K$  and  $K' \models \text{B}\Sigma_1 + \text{exp}$ .

*Proof.* Pick any  $b \in K \setminus M$  such that  $2^{(b,b)}$  exists in  $K$ . Consider

$$K' = \log^{\mathbb{N}}(b) = \inf\{\log^{(i)}(b) \in K : i \in \mathbb{N}\}.$$

As alluded to in the proof of Corollary 4.8, the implication (c)  $\Rightarrow$  (a) in Corollary 4.9 does not need  $(M, \mathcal{X})$  to be countably generated. Therefore, we are done if  $M \neq \mathbb{N}$ . So suppose  $M = \mathbb{N}$ . Then  $K \models \log^{(i)}(b) > i$  for every  $i \in \mathbb{N}$ . Thus, by  $\Delta_0$  overspill in  $K$ , we see that  $\mathbb{N} = M \subsetneq_e K'$ .  $\square$

Kaufmann noticed that the Arithmetized Completeness Theorem can be used to show the Mac Dowell–Specker Theorem [54]. Adapting his ideas to our setting gives us an alternative proof of a recent theorem of Schmerl's [55].

In the proof, we will need an  $\mathcal{L}_A$  formula  $\Pi_n\text{-Sat}(x)$  for the satisfaction of  $\Pi_n$  formulas for each  $n \in \mathbb{N}$ . These formulas are required to satisfy the following.

- (1)  $\text{I}\Delta_0 + \text{exp} \vdash \forall x (\Pi_n\text{-Sat}(x) \rightarrow x \in \Pi_n)$ .
- (2)  $\text{I}\Delta_0 + \text{exp}$  proves that  $\Pi_n\text{-Sat}$  satisfies Tarski's inductive clauses for satisfaction for  $\Pi_n$  formulas.
- (3)  $\text{I}\Delta_0 + \text{exp} \vdash \forall a (\Pi_n\text{-Sat}(\ulcorner \theta(\check{a}) \urcorner) \leftrightarrow \theta(a))$  for all  $\theta(x) \in \Pi_n$ .
- (4)  $\text{I}\Delta_0 + \text{exp} \vdash \forall \sigma \in \Pi_n (\Pi_n\text{-Sat}(\sigma) \leftrightarrow \Pi_{n+1}\text{-Sat}(\sigma))$ .
- (5) If  $n = 0$ , then  $\Pi_n\text{-Sat}$  is  $\Delta_1$  over  $\text{I}\Delta_0 + \text{exp}$ ; otherwise, it is  $\Pi_n$ .

Notice property (2) implies property (3). For details on how these formulas can be constructed, consult Section V.5(b) of [32] or Section 7.2 of [70].

The  $\Pi_n\text{-Sat}$ 's enable one to axiomatize PA in terms of uniform reflection.

**Theorem 4.11** (Mostowski, Kreisel–Lévy). PA is equivalent over  $\text{I}\Delta_0 + \text{exp}$  to

$$\{\forall \tau (\Pi_{n+1}\text{-Sat}(\tau) \rightarrow \text{Con}(\tau)) : n \in \mathbb{N}\}.$$

*Proof.* See Theorem 5.1 in Beklemishev [10], which refines the work by Mostowski [46] and Kreisel–Lévy [42].  $\square$

Schmerl [55] originally formulated his generalization of the Mac Dowell–Specker Theorem using the parameter-free arithmetical comprehension scheme  $\text{ACA}_0^-$ . Let us give here an equivalent formulation in terms of definable sets.



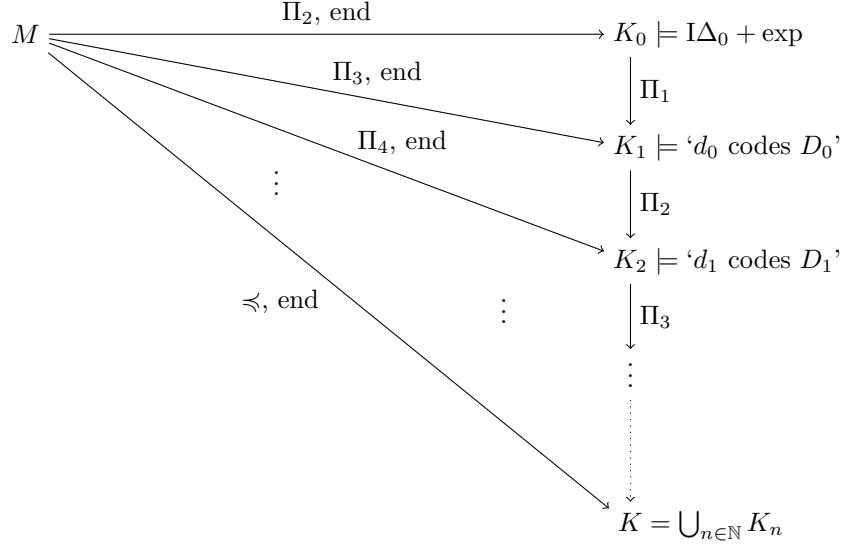


FIGURE 1. Elementarily end extending  $M$  to  $K$  such that  $\text{SSy}_M(K)$  is  $\Delta_1^0$ -generated by  $\{D_n : n \in \mathbb{N}\}$

**Theorem 4.12** (Schmerl). Let  $(M, \mathcal{X})$  be a countably generated model of  $\text{WKL}_0^*$  in which  $\mathcal{X} \supseteq \text{Def}(M)$ . Then there is  $K \succeq_e M$  such that  $\text{SSy}_M(K) = \mathcal{X}$ .

*Proof.* First, notice  $M \models \text{PA}$ . Second, observe that whenever  $n \in \mathbb{N}$ , the set  $S_n$  defined by

$$S_n = \{\sigma \in M : M \models \Pi_n\text{-Sat}(\sigma)\}$$

is parameter-free definable in  $M$ , and so must be in  $\mathcal{X}$  by assumption.

Let  $(D_n)_{n \in \mathbb{N}}$  enumerate a subset of  $\mathcal{X}$  that  $\Delta_1^0$ -generates  $(M, \mathcal{X})$ . We will iteratively find consistent complete Henkinized theories  $T_0, T_1, T_2, \dots$  in  $(M, \mathcal{X})$  such that for every  $n \in \mathbb{N}$ ,

- the language of  $T_n$  consists of the symbols in  $\mathcal{L}_A^M$  and some extra constant symbols;
- $T_n \supseteq S_{n+2}$ ;
- $T_{n+1} \supseteq \Pi_{n+1}^M \cap T_n$ , where  $\Pi_{n+1}^M \cap T_n$  denotes the set of all formulas from  $T_n$  with  $\Pi_{n+1}^M$  shapes; and
- $T_{n+1} \supseteq \{\check{m} \in_{\text{Ack}} d : m \in D_n\} \cup \{\check{m} \notin_{\text{Ack}} d : m \in M \setminus D_n\}$  for some constant symbol  $d$ .

Each  $T_n$  has a Henkin model  $K_n$ . Our conditions on the  $T_n$ 's ensure that for every  $n \in \mathbb{N}$ ,

$$M \preceq_{\Pi_{n+2}} K_n \preceq_{\Pi_{n+1}} K_{n+1} \quad \text{and} \quad D_n \in \text{SSy}_M(K_{n+1}),$$

as illustrated in Figure 1. Using Theorem 4.11 and Corollary 3.3, pick any consistent complete Henkinized theory  $T_0 \supseteq S_2$  in  $(M, \mathcal{X})$ . Given  $T_n$  satisfying the inductive conditions, we similarly find a consistent complete Henkinized theory  $T_{n+1}$  extending

$$\begin{aligned} & S_{n+3} \cup (\Pi_{n+1}^M \cap T_n) \\ & \cup \{\check{m} \in_{\text{Ack}} d_n : m \in D_n\} \\ & \cup \{\check{m} \notin_{\text{Ack}} d_n : m \in M \setminus D_n\} \end{aligned}$$

in  $(M, \mathcal{X})$ , where  $d_n$  is a constant symbol new to the language of  $T_n$ . The arguments in the proof of Proposition 4.4 show that such  $T_{n+1}$  exists provided  $(M, \mathcal{X}) \models$

$\text{Con}(S_{n+3} \cup (\Pi_{n+1} \cap T_n))$ . So, in view of Lemma 3.1, it suffices to prove  $M \models \text{Con}(\sigma \wedge \tau)$  for all  $\sigma \in S_{n+3}$  and  $\tau \in \Pi_{n+1}^M \cap T_n$ .

Pick any  $\sigma \in S_{n+3}$  and  $\tau \in \Pi_{n+1}^M \cap T_n$ . Let  $\bar{c}$  list all the constant symbols in  $\tau$  that are not in  $\mathcal{L}_A^M$ , and write  $\tau$  as  $\tau(\bar{c})$ . (The length of the tuple  $\bar{c}$  may be a nonstandard element of  $M$ .) Then  $T_n$  contains the  $\Sigma_{n+2}^M$  sentence  $\exists \bar{w} \tau(\bar{w})$  because it is consistent and complete in  $(M, \mathcal{X})$ . Since  $S_{n+2} \subseteq T_n$  and  $(M, \mathcal{X}) \models \text{Con}(T_n)$ , this implies  $\ulcorner \forall \bar{w} \neg \tau(\bar{w}) \urcorner \notin S_{n+2}$ . Property (4) of  $\Pi_{n+2}$ -Sat and property (2) of  $\Pi_{n+3}$ -Sat then tell us  $M \models \Pi_{n+3}\text{-Sat}(\sigma \wedge \exists \bar{w} \tau(\bar{w}))$ . Hence  $M \models \text{Con}(\sigma \wedge \exists \bar{w} \tau(\bar{w}))$  by Theorem 4.11. As the constant symbols  $\bar{c}$  do not appear in  $\sigma$ , we conclude  $M \models \text{Con}(\sigma \wedge \tau(\bar{c}))$ , as required.

At the end, the model we want is  $K = \bigcup_{n \in \mathbb{N}} K_n$ . This is an end extension of  $M$  because all the  $K_n$ 's are, as the proof of Proposition 3.4 shows. This extension is proper because some of the  $D_n$ 's must be unbounded in  $M$ .  $\square$

Schmerl [55, Corollary 7] used this and Theorem 2.12 to prove a conjecture in Kossak–Paris [40, page 71] about coding  $n^+$ -inductive sets in elementary end extensions. In view of Theorem 2.11, his argument actually also proves Kossak and Paris's conjecture about  $n^+$ -collective sets [40, page 71]:

**Corollary 4.13.** Fix a countable  $M \models \text{PA}$  and  $n \in \mathbb{N}$ . If  $X \subseteq M$  such that  $(M, \text{Def}(M) \cup \{X\}) \models \text{B}\Sigma_{n+1}^0$ , then there is  $K \succeq_e M$  in which  $X \in \text{SSy}_M(K)$  and  $(M, \text{SSy}_M(K)) \models \text{B}\Sigma_{n+1}^0$ .  $\square$

Another application of Theorem 4.12 is in establishing variants of Corollary 4.8 and Corollary 4.9 for elementary end extensions, the details of which the reader can easily fill in.

We next prove a generalization of a theorem of Wilkie's [72] using the method from Lessan [43, Theorem 2.2.20] and Schmerl [54].

**Theorem 4.14.** Let  $(M, \mathcal{X})$  be a countably generated model of  $\text{WKL}_0^*$  in which  $M$  is nonstandard and short  $\Pi_1$ -recursively saturated. Let  $a \in M$  and  $\Phi(z)$  be a complete set of  $\mathcal{L}_A$  formulas extending PA. The following are equivalent.

- (a)  $\Pi_n \cap \Phi \in \text{SSy}(M)$  for every  $n \in \mathbb{N}$ , and  $M \models \Pi_1\text{-Th}(\Phi(a))$ .
- (b) There is  $K \models \Phi(a)$  properly end extending  $M$  such that  $\text{SSy}_M(K) = \mathcal{X}$ .

*Proof.* The implication (b)  $\Rightarrow$  (a) is clear. For the converse, suppose (a) is true. For each  $n \in \mathbb{N}$ , pick  $c_n \in M$  that codes  $\Pi_n \cap \Phi$ . Let  $(D_n)_{n \in \mathbb{N}}$  enumerate a subset of  $\mathcal{X}$  which  $\Delta_1^0$ -generates  $(M, \mathcal{X})$ . We will iteratively find consistent complete Henkinized theories  $T_0, T_1, T_2, \dots$  in  $(M, \mathcal{X})$  such that for every  $n \in \mathbb{N}$ ,

- the language of  $T_n$  consists of the symbols in  $\mathcal{L}_A^M$  and some extra constant symbols;
- $T_n \supseteq \Pi_{n+2} \cap \Phi(\hat{a})$ , where  $\Pi_{n+2} \cap \Phi(\hat{a})$  denotes the set of all formulas from  $\Phi(\hat{a})$  with  $\Pi_{n+2}$  shapes;
- $T_{n+1} \supseteq \Pi_{n+1}^M \cap T_n$ ; and
- $T_{n+1} \supseteq \{\check{m} \in_{\text{Ack}} d : m \in D_n\} \cup \{\check{m} \notin_{\text{Ack}} d : m \in M \setminus D_n\}$  for some constant symbol  $d$ .

Each  $T_n$  has a Henkin model  $K_n$ . All these Henkin models can be viewed as end extensions of  $M$  as in the proof of Proposition 3.4. Moreover, our conditions on the  $T_n$ 's make

$$K_{n+1} \succeq_{\Pi_{n+1}} K_n \models \Pi_{n+2}\text{-Th}(\Phi(a)) \quad \text{and} \quad D_n \in \text{SSy}_M(K_{n+1})$$

for every  $n \in \mathbb{N}$ . It is not hard to see that this is sufficient to imply (b).

Since  $M \models \Pi_1\text{-Th}(\Phi(a))$ , we know  $M \models \text{Con}(\varphi(\hat{a}))$  whenever  $\varphi \in \Pi_2 \cap \Phi$  by Theorem 4.11. Short  $\Pi_1$  recursive saturation then gives us a nonstandard  $\varphi_2 \in_{\text{Ack}}^M c_2$  such that  $M \models \text{Con}(\varphi_2(\hat{a}))$ . Additionally, we make  $M \models \text{Prov}(\varphi_2, \varphi)$  for all

standard  $\varphi \in \Pi_2 \cap \Phi$ . This can be achieved using a coded sequence of formulas from  $\Pi_2 \cap \Phi$  indexed by  $\mathbb{N}$  such that (modulo simple syntactical manipulations) each element of the sequence contains all earlier ones as conjuncts, and every formula in  $\Pi_2 \cap \Phi$  appears as a conjunct in some element of the sequence. One can construct such a sequence by repeatedly taking conjunctions of formulas from  $\Pi_2 \cap \Phi$ , for example. Now apply Corollary 3.3 to find a theory  $T_0 \in \mathcal{X}$  that contains  $\varphi_2(\check{a})$  and is consistent, complete, and Henkinized in the sense of  $(M, \mathcal{X})$ .

Suppose  $T_n$  is already found satisfying the inductive conditions. If  $\varphi \in \Pi_{n+3} \cap \Phi$ , then the formula

$$\forall \tau \in \Pi_{n+1} (\Pi_{n+1}\text{-Sat}(\tau) \rightarrow \text{Con}(\varphi(\check{a}) \wedge \tau))$$

is  $\Pi_{n+2}$  by property (5) of  $\Pi_{n+1}\text{-Sat}$ , and so it must be in  $\Pi_{n+2}\text{-Th}(\Phi(a))$  by Theorem 4.11. Apply  $\Sigma_0^0$  overspill in  $(M, \mathcal{X})$  and the trick in the previous paragraph to find a  $\Pi_{n+3}^M$  formula  $\varphi_{n+3} \in_{\text{Ack}}^M c_{n+3}$  such that for all standard  $\varphi \in \Pi_{n+3} \cap \Phi$ ,

$$(M, \mathcal{X}) \models \ulcorner \forall \tau \in \Pi_{n+1} (\Pi_{n+1}\text{-Sat}(\tau) \rightarrow \text{Con}(\varphi_{n+3}(\check{a}) \wedge \tau)) \urcorner \in T_n \wedge \text{Prov}(\varphi_{n+3}, \varphi).$$

We show  $(M, \mathcal{X}) \models \text{Con}(\{\varphi_{n+3}(\check{a})\} \cup (\Pi_{n+1} \cap T_n))$ . Our theory  $T_{n+1}$  will then be some consistent complete Henkinized theory in  $(M, \mathcal{X})$  extending

$$\begin{aligned} & \{\varphi_{n+3}(\check{a})\} \cup (\Pi_{n+1}^M \cap T_n) \\ & \cup \{\check{m} \in_{\text{Ack}} d_n : m \in D_n\} \\ & \cup \{\check{m} \notin_{\text{Ack}} d_n : m \in M \setminus D_n\} \end{aligned}$$

given by Corollary 3.3.

In view of Lemma 3.1, it suffices to prove that  $M \models \text{Con}(\varphi_{n+3}(\check{a}) \wedge \tau)$  for all  $\tau \in \Pi_{n+1}^M \cap T_n$ . Take any such  $\tau$ . Let  $\bar{c}$  list all constant symbols appearing in  $\tau$  that are not in  $\mathcal{L}_A^M$ . (The length of the tuple  $\bar{c}$  may be a nonstandard element of  $M$ .) Write  $\tau$  as  $\tau(\bar{c})$ . As shown in Pudlák [51, Section 5], property (3) of  $\Pi_{n+1}\text{-Sat}$  is provable in  $\text{WKL}_0^*$ . Therefore, as  $T_n \supseteq \Pi_{n+2} \cap \Phi(\check{a}) \supseteq \text{I}\Delta_0 + \text{exp}$ , we know that  $T_n$  contains  $\Pi_{n+1}\text{-Sat}(\tau(\bar{c}))$ , where  $\bar{c}$  denotes the term  $0 + 1 + 1 + \dots + 1$  in which the number of 1's is equal to the value of  $c$  as a constant (not as an element of  $M$ ). Our choice of  $\varphi_{n+3}$  then tells us  $T_n$  contains  $\text{Con}(\varphi_{n+3}(\check{a}) \wedge \tau(\bar{c}))$  and hence also  $\text{Con}(\varphi_{n+3}(\check{a}) \wedge \exists \bar{w} \tau(\bar{w}))$ . So  $K_n \models \text{Con}(\varphi_{n+3}(\check{a}) \wedge \exists \bar{w} \tau(\bar{w}))$  because  $\check{a}^{K_n} = a$  and the formula  $\text{Con}(\cdot)$  has standard shape. Thus, the fact that  $\ulcorner \varphi_{n+3}(\check{a}) \wedge \exists \bar{w} \tau(\bar{w}) \urcorner \in M \subseteq_e K_n$  implies  $M \models \text{Con}(\varphi_{n+3}(\check{a}) \wedge \exists \bar{w} \tau(\bar{w}))$ . Since  $\bar{c}$  do not appear in  $\varphi_{n+3}(\check{a})$ , we conclude  $M \models \text{Con}(\varphi_{n+3}(\check{a}) \wedge \tau(\bar{c}))$ .  $\square$

**Corollary 4.15.** Let  $M$  be a nonstandard short  $\Pi_1$ -recursively saturated  $\text{WKL}_0^*$ -expandable model of  $\text{B}\Sigma_1 + \text{exp}$ . Let  $a \in M$  and  $\Phi(z)$  be a complete set of  $\mathcal{L}_A$  formulas extending PA. The following are equivalent.

- (a)  $\Pi_n \cap \Phi \in \text{SSy}(M)$  for every  $n \in \mathbb{N}$ , and  $M \models \Pi_1\text{-Th}(\Phi(a))$ .
- (b) There is  $K \models \Phi(a)$  properly end extending  $M$ .

*Proof.* Since  $\text{Con}(\text{I}\Delta_0) \in \Pi_1\text{-Th}(\text{PA}) \subseteq \Pi_1\text{-Th}(\Phi(a))$  and  $M \models \Pi_1\text{-Th}(\Phi(a))$ , we know  $M \models \text{Con}(\text{I}\Delta_0)$ . So  $M$  is expandable to a countably generated  $(M, \mathcal{X}) \models \text{WKL}_0^*$  by Corollary 4.5. This justifies the applicability of Theorem 4.14. Alternatively, use the construction in the proof of Theorem 4.14 directly.  $\square$

**Question 4.16.** Can the requirement on  $\Phi$  in being an extension of PA in Theorem 4.14 be relaxed?

With only condition (a) in Theorem 4.14, it is not possible to guarantee a recursively saturated end extension satisfying  $\Phi$ , because every recursively saturated model of  $\text{I}\Delta_0$  codes its own complete theory. As observed by Smoryński [63,

page 276], this is the only obstacle. The next theorem extends Theorem 3.5 in [63] and refines part of Theorem 2.3 in McAloon [45].

**Theorem 4.17.** Let  $M$  be a nonstandard short  $\Pi_1$ -recursively saturated  $\text{WKL}_0^*$ -expandable model of  $\text{B}\Sigma_1 + \text{exp}$ . Let  $a \in M$  and  $\Phi(z)$  be a complete set of  $\mathcal{L}_A$  formulas extending PA. The following are equivalent.

- (a)  $\Phi \in \text{SSy}(M)$  and  $M \models \Pi_1\text{-Th}(\Phi(a))$ .
- (b) There is a recursively saturated  $K \models \Phi(a)$  that properly end extends  $M$ .

*Proof sketch.* Assume (a) holds. For every  $\varphi \in \Phi$ , we have  $\Phi(z) \vdash \text{Con}(\varphi(\check{z}))$  by Theorem 4.11, and so  $M \models \text{Con}(\varphi(\check{a}))$  by (a). Thanks to the codedness of  $\Phi$  and short  $\Pi_1$  recursive saturation, this overflows. An application of Proposition 3.4 then proves (b).  $\square$

The next result is motivated by Theorem 4.14 and Theorem 4.17.

**Theorem 4.18.** Let  $(M, \mathcal{X})$  be a countably generated model of  $\text{WKL}_0^*$  in which  $M$  is short  $\Pi_1$ -recursively saturated and nonstandard. Let  $a \in M$  and  $\Phi(z)$  be a complete set of  $\mathcal{L}_A$  formulas extending PA. The following are equivalent.

- (a)  $\Phi \in \text{SSy}(M)$  and  $M \models \Pi_1\text{-Th}(\Phi(a))$ .
- (b) There is a recursively saturated  $K \models \Phi(a)$  properly end extending  $M$  such that  $\text{SSy}_M(K) = \mathcal{X}$ .

*Proof sketch.* Start by imitating the proof of Theorem 4.17. Then use Corollary 3.3 repeatedly to build an increasing elementary chain of recursively saturated models of  $\Phi(a)$  that code more and more sets in  $\mathcal{X}$ . The union of this chain is what we want. The argument is similar to, and slightly simpler than, those in the proofs of Theorem 4.12 and Theorem 4.14.  $\square$

Notice the completeness of  $\Phi$  is not needed to show the implications (a)  $\Rightarrow$  (b) in Theorem 4.14, Corollary 4.15, Theorem 4.17 and Theorem 4.18. If we start off with  $\Phi \in \text{SSy}(M)$ , then it is also not needed for the converse implications. One can thus devise a wide range of variants of Theorem 4.17 and Theorem 4.18, amongst which the following is the most relevant for us; see Corollary 4.23.

**Lemma 4.19.** Let  $(M, \mathcal{X})$  be a countably generated model of  $\text{WKL}_0^*$  in which  $M$  is nonstandard and short  $\Pi_1$ -recursively saturated. Let  $a \in M$  and  $\Phi(z)$  be a set of  $\Pi_1$  formulas extending  $\text{I}\Delta_0$  in  $\text{SSy}(M)$ . The following are equivalent.

- (a)  $M \models \Phi(a)$ .
- (b) There is  $K \models \Phi(a)$  properly end extending  $M$  such that  $\text{SSy}_M(K) = \mathcal{X}$ .

*Proof.* Suppose (a) holds. Take  $c \in M$  that codes  $\Phi$ . Theorem 3.7 implies  $M \models \text{TabCon}(\varphi(\check{a}))$  for all standard  $\varphi \in \overset{M}{\text{Ack}} c$ . Therefore, by short  $\Pi_1$  recursive saturation (and the trick involving conjunctions in the proof of Theorem 4.14), one finds a  $\Pi_1^M$  formula  $\psi \in \overset{M}{\text{Ack}} c$  such that

$$M \models \text{TabCon}(\psi(\check{a})) \wedge \text{TabProv}(\psi(\check{a}), \varphi(\check{a}))$$

for all  $\varphi \in \Phi$ . We now get the extension for (b) by applying the proof of Theorem 4.6 to  $\psi(\check{a})$ .  $\square$

As the reader can check, in Theorem 4.17 and Theorem 4.18, if  $\Phi$  is a set of sentences, then it need not extend PA — having local reflection on top of  $\text{I}\Delta_0 + \text{exp}$  is enough.

The next result shows that a much more general statement is true if the model is outright countable; cf. Guaspari [30, Theorem 6.5].

**Theorem 4.20.** Let  $M$  be a countable nonstandard short  $\Pi_1$ -recursively saturated model of  $\text{BSy}_1$  and  $a \in M$ . Let  $\mathcal{L}_A^*$  be a recursive language extending  $\mathcal{L}_A$  and  $\Phi(z)$  be a set of  $\mathcal{L}_A^*$  formulas in  $\text{SSy}(M)$ . The following are equivalent.

- (a)  $M \models \Pi_1\text{-Th}(\Phi(a))$ .
- (b) Some proper end extension of  $M$  expands to a recursively saturated model of  $\Phi(a)$ .

*Proof.* The ideas come from two arguments in the literature: one due to Paris and Kirby for showing a similar statement [50, Theorem C], and another due to Ressayre for showing the resplendency of countable recursively saturated models [53, Theorem 2.3]; see also [35, Theorem 15.7].

Suppose  $M \models \Pi_1\text{-Th}(\Phi(a))$ . Add to  $\mathcal{L}_A^*$  one new constant symbol for each element of  $M$ , and another countably many new constant symbols to be used for Henkinization. Call the resulting language  $\mathcal{H}$ .

By recursion, we will find sets of  $\mathcal{H}$  sentences

$$\Phi(a) = \Phi_0 \subseteq \Phi_1 \subseteq \Phi_2 \subseteq \dots$$

such that  $\Phi_\omega = \bigcup_{n \in \mathbb{N}} \Phi_n$  is consistent, complete, and Henkinized. The construction will be designed to make the reduct  $K$  of the Henkin model of  $\Phi_\omega$  to  $\mathcal{L}_A$  the extension we want. During the construction, we assume inductively that for every  $n \in \mathbb{N}$ ,

- (i)  $\Phi_n \in \text{SSy}(M)$ ;
- (ii)  $\Phi_n$  mentions only finitely many parameters  $\bar{a} \in M$  and finitely many Henkin constants; and
- (iii)  $M \models \Pi_1\text{-Th}(\Phi_n)$ , in the sense that  $M \models \xi(\bar{a})$  whenever  $\Phi_n(\bar{a}) \vdash \xi(\bar{a})$  and  $\xi \in \Pi_1$ .

By our hypotheses, we know  $\Phi_0$  satisfies these conditions.

First, let us show how to ensure recursive saturation. This will, in particular, make  $\Phi_\omega$  Henkinized and complete. Suppose  $\Phi_n$  is found. Consider a recursive set of  $\mathcal{H}$  formulas  $p(v)$  that mentions only finitely many Henkin constants and finitely many parameters from  $M$ . Take any Henkin constant  $c_0$  new to  $\Phi_n$  and  $p(v)$ . If setting  $\Phi_{n+1} = \Phi_n \cup p(c_0)$  preserves the inductive conditions, then we are already done. So suppose not. Let  $\psi(v)$  be a conjunction of formulas from  $p(v)$  and  $\xi \in \Pi_1$  possibly involving some parameters from  $M$  mentioned in  $\Phi_n \cup p(v)$  such that

$$\Phi_n \cup \{\psi(c_0)\} \vdash \xi \quad \text{and} \quad M \models \neg\xi.$$

It suffices to show  $\Phi_{n+1} = \Phi_n \cup \{\forall v \neg\psi(v)\}$  again satisfies the inductive conditions. Take any  $\zeta \in \Pi_1\text{-Th}(\Phi_{n+1})$ . Since  $c_0$  does not appear in  $\Phi_n$  and in  $\xi$ , actually  $\Phi_n \cup \{\exists v \psi(v)\} \vdash \xi$ . Thus  $\Phi_n \vdash \xi \vee \zeta$ . Notice  $\xi \vee \zeta$  may involve parameters  $\bar{a}' \in M$  that do not appear in  $\Phi_n$ . So let us write  $\xi$  and  $\zeta$  as  $\xi(\bar{a}')$  and  $\zeta(\bar{a}')$  respectively. Then  $\Phi_n \vdash \forall \bar{w} (\xi(\bar{w}) \vee \zeta(\bar{w}))$  as  $\bar{a}'$  does not appear in  $\Phi_n$ . Since  $\forall \bar{w} (\xi(\bar{w}) \vee \zeta(\bar{w}))$  is  $\Pi_1$ , condition (iii) implies  $M \models \forall \bar{w} (\xi(\bar{w}) \vee \zeta(\bar{w}))$ . Hence  $M \models \zeta(\bar{a}')$ , as required.

With (iii), we can view  $M$  as a substructure of  $K$  in a natural way. Let us show how to make  $M \subseteq_e K$ . Suppose  $\Phi_n$  is found. Let  $a_0, a_1, \dots, a_\ell$  be the parameters from  $M$  occurring in  $\Phi_n$ , and  $c$  be a Henkin constant such that  $\Phi_n(\bar{a}) \vdash c < a_0$ . If we can set  $\Phi_{n+1} = \Phi_n \cup \{c = a_i\}$  for some  $i \leq \ell$ , then we are done. So suppose not. Assume further that  $\Phi_n(\bar{a})$  contains  $c \neq a_i$  for every  $i \leq \ell$ . As the previous paragraph shows, this further assumption does not cause any loss of generality. Consider

$$q(v) = \{\xi(v, \bar{a}) : \xi \in \Pi_1 \text{ and } \Phi_n(\bar{a}) \vdash \xi(c, \bar{a})\}.$$

Notice the formula  $v < a_0$  is in  $q(v)$ , and  $q(v)$  is logically equivalent to a set of  $\Pi_1$  formulas in  $\text{SSy}(M)$  by Craig's Trick. We claim that  $q(v)$  is finitely satisfiable

in  $M$ . Take  $\xi \in \Pi_1$  and suppose  $\Phi_n(\bar{a}) \vdash \xi(c, \bar{a})$ . Recall  $\Phi_n(\bar{a}) \vdash c < a_0$  too. So writing  $\xi(v, \bar{a})$  as  $\forall \bar{x} \zeta(v, \bar{x}, \bar{a})$ , where  $\zeta \in \Delta_0$ , we have

$$\begin{aligned} & \Phi_n(\bar{a}) \vdash \exists v < a_0 \forall \bar{x} \zeta(v, \bar{x}, \bar{a}) \\ \therefore & \Phi_n(\bar{a}) \vdash \forall b \exists v < a_0 \forall \bar{x} < b \zeta(v, \bar{x}, \bar{a}) && \text{by pure logic,} \\ \therefore & M \models \forall b \exists v < a_0 \forall \bar{x} < b \zeta(v, \bar{x}, \bar{a}) && \text{by condition (iii),} \\ \therefore & M \models \exists v < a_0 \forall \bar{x} \zeta(v, \bar{x}, \bar{a}) && \text{since } M \models \text{B}\Sigma_1. \end{aligned}$$

Hence  $M \models \exists v < a_0 \xi(v, \bar{a})$ , as claimed.

Apply short  $\Pi_1$  recursive saturation to find  $m \in M$  realizing  $q$ . Notice  $m \notin \{a_0, a_1, \dots, a_\ell\}$  because the formula  $v \neq a_i$  is in  $q(v)$  for each  $i \leq \ell$ . Set  $\Phi_{n+1} = \Phi_n \cup \{c = m\}$ . If  $\Phi_{n+1}(m, \bar{a}) \vdash \xi(m, \bar{a})$ , where  $\xi \in \Pi_1$ , then  $\Phi_n(\bar{a}) \vdash \forall v (c = v \rightarrow \xi(v, \bar{a}))$  as  $m \notin \{a_0, a_1, \dots, a_\ell\}$ , and so  $\xi(v, \bar{a}) \in q(v)$ , making  $M \models \xi(m, \bar{a})$ . Therefore, the inductive conditions are again satisfied.

At the end, if  $K \neq M$ , then the proof is already finished. So suppose not. Then  $M$  is itself recursively saturated. By a self-embedding theorem of Solovay's [48, Theorem 4], this implies the existence of an embedding  $f: M \rightarrow M$  fixing  $a$  for which  $\text{Im}(f) \subsetneq_e M$ . Hence we are done as well. (See Proposition 5.1 in the next section for an improved version of Solovay's theorem.)  $\square$

*Remark 4.21.* Theorem 4.20 generalizes naturally to every level of the arithmetic hierarchy: if  $n \in \mathbb{N}$ , then we can make the end extension  $\Pi_n$ -elementary in (b) by requiring  $M$  to be short  $\Pi_{n+1}$ -recursively saturated and satisfy  $\text{B}\Sigma_{n+1} + \Pi_1(\Sigma_n)\text{-Th}(\Phi(a))$ . Here  $\Pi_1(\Sigma_n)$  denotes the set of formulas of the form  $\forall \bar{v} \zeta(\bar{v}, \bar{z})$ , where  $\zeta \in \Delta_0(\Sigma_n)$ . Induction is only used to guarantee that  $\text{SSy}(M)$  is closed under relative recursion. So, for instance, if  $\Phi$  is recursive, then one can actually replace  $\text{B}\Sigma_{n+1}$  with the induction-free  $\Sigma_{n+1}$  collection scheme here. Alternatively, Theorem 4.20 can be proved using an argument from Blanck–Enayat [13] if we additionally assume  $\Phi \supseteq \text{PA}^-$  and  $\Pi_1\text{-Th}(M, a') \in \text{SSy}(M)$  for all  $a' \in M$ .

As noticed by Paris [21, pages 224–225], the saturation condition on  $M$  in Theorem 4.20 cannot be omitted in general, even when  $\Phi = \text{PA}$  and  $M \models \text{exp}$ , and when the saturation condition on the end extension is ignored. We next show that short  $\Pi_1$  recursive saturation is actually necessary here.

**Corollary 4.22.** The following are equivalent for a countable  $M \models \text{I}\Delta_0$ .

- (a)  $M \models \text{B}\Sigma_1$  and is short  $\Pi_1$ -recursively saturated.
- (b) Whenever  $a \in M$  and  $\Phi(z)$  is a recursive set of formulas in a finite language extending  $\mathcal{L}_A$ , if  $M \models \mathcal{L}_A\text{-Th}(\Phi(a))$ , then there is a proper end extension  $K \supsetneq_e M$  that expands to a model of  $\Phi(a)$ .

*Proof.* The interesting case is when  $M \neq \mathbb{N}$ , for which we have already proved one direction. For the converse, suppose (b) holds. Setting  $\Phi = \text{I}\Delta_0$  tells us  $M \models \text{B}\Sigma_1$ . Pick any  $\Pi_1$  type

$$p(v, a) = \{\varphi_i(v, a) : i \in \mathbb{N}\} \cup \{v < a\}$$

over  $M$  where  $a \in M$  and the enumeration  $(\varphi_i)_{i \in \mathbb{N}}$  is recursive. Let  $c$  be a constant symbol new to  $\mathcal{L}_A$ . Define  $\Phi(a) = p(c, a)$ . Notice  $M \models \mathcal{L}_A\text{-Th}(\Phi(a))$  because  $p(v, a)$  is finitely satisfiable in  $M$ . Apply (b) to get  $K \supsetneq_e M$  in which  $p$  is realized. Let  $c$  denote a realization of  $p$  in  $K$ . Since  $c < a \in M \subseteq_e K$ , we know  $c \in M$ . Since  $M \subseteq_e K$  and each  $\varphi_i \in \Pi_1$ , we know  $M \models p(c, a)$ .  $\square$

Kotlarski [63, page 279] used an argument similar to our proof of Theorem 4.20 to show that a countable model of PA is short recursively saturated if and only if it has a recursively saturated elementary end extension. One can view Corollary 4.22 as the level-one case of a hierarchical version of this fact.

Given what we have established, it is not hard to put together a second-order version of Theorem 4.20.

**Corollary 4.23.** Let  $(M, \mathcal{X})$  be a countable model of  $\text{WKL}_0^*$  and  $a \in M$  such that  $M$  is nonstandard and short  $\Pi_1$ -recursively saturated. Let  $\mathcal{L}_A^*$  be a recursive language extending  $\mathcal{L}_A$  and  $\Phi(z)$  be a set of  $\mathcal{L}_A^*$  formulas extending  $\text{I}\Delta_0$  in  $\text{SSy}(M)$ . The following are equivalent.

- (a)  $M \models \Pi_1\text{-Th}(\Phi(a))$ .
- (b)  $M$  has a proper end extension  $K$  which expands to a recursively saturated model of  $\Phi(a)$  and makes  $\text{SSy}_M(K) = \mathcal{X}$ .

*Proof.* Notice  $\Pi_1\text{-Th}(\Phi(z))$  is logically equivalent to a set of  $\Pi_1$  formulas in  $\text{SSy}(M)$  by Craig's Trick. Apply Lemma 4.19 to find  $M' \models \Pi_1\text{-Th}(\Phi(a))$  properly end extending  $M$  such that  $\text{SSy}_M(M') = \mathcal{X}$ . In view of Corollary 2.10, we may additionally assume  $M'$  is a recursively saturated model of  $\text{B}\Sigma_1$ . Then Theorem 4.20 gives us the end extension  $K$  we want.  $\square$

Theorem 4.20 enables us to build  $\Sigma_1$ -closed end extensions. Recall a structure  $M$  is  $\Sigma_1$ -closed for an  $\mathcal{L}_A$  theory  $T$  if  $M \models T$  and for every  $K \models T$ ,

$$M \preceq_{\Delta_0} K \quad \Rightarrow \quad M \preceq_{\Pi_1} K.$$

Up-to-date information about  $\Sigma_1$ -closed models of arithmetic can be found in a recent paper by Adamowicz, Cordón-Franco, and Lara-Martín [7]. As the reader can verify, the theorem below remains true if we remove both instances of  $\text{exp}$  from the statement; the proof is more or less the same.

**Theorem 4.24.** Every countable short  $\Pi_1$ -recursively saturated  $M \models \text{B}\Sigma_1 + \text{exp}$  has a proper end extension that is  $\Sigma_1$ -closed for  $\text{I}\Delta_0 + \text{exp}$ .

*Proof.* We construct countable short  $\Pi_1$ -recursively saturated models of  $\text{B}\Sigma_1 + \text{exp}$

$$M = K_0 \subseteq_e K_1 \subseteq_e K_2 \subseteq_e \dots$$

by recursion. Suppose  $K_n$  is found. Let  $\varphi \in \Sigma_1$  and  $a \in K_n$  such that  $K_n \models \Pi_1\text{-Th}(\text{I}\Delta_0 + \text{exp} + \varphi(a))$ . Proposition 2.4 then says actually  $K_n \models \Pi_1\text{-Th}(\text{B}\Sigma_1 + \text{exp} + \varphi(a))$ . So we can carry on the construction with a countable short  $\Pi_1$ -recursively saturated  $K_{n+1} \models \text{B}\Sigma_1 + \text{exp} + \varphi(a)$  properly end extending  $K_n$  given by Theorem 4.20. Since all the models involved are countable, one can arrange the  $\varphi$ 's and the  $a$ 's such that  $\bigcup_{n \in \mathbb{N}} K_n$  is  $\Sigma_1$ -closed for  $\text{I}\Delta_0 + \text{exp}$  at the end.  $\square$

The theorem above says that the first-order theory of proper cuts closed under exponentiation in structures  $\Sigma_1$ -closed for  $\text{I}\Delta_0 + \text{exp}$  is exactly  $\text{B}\Sigma_1 + \text{exp}$ . We also have a second-order counterpart of this.

**Corollary 4.25.** For every countable short  $\Pi_1^0$ -recursively saturated  $(M, \mathcal{X}) \models \text{WKL}_0^*$ , there exists a proper end extension  $K \supseteq_e M$  which is  $\Sigma_1$ -closed for  $\text{I}\Delta_0 + \text{exp}$  such that  $\text{SSy}_M(K) = \mathcal{X}$ .

*Proof.* Use Theorem 4.6, Corollary 4.10, and Theorem 2.9 to find a countable recursively saturated  $M' \models \text{B}\Sigma_1 + \text{exp}$  properly end extending  $M$  such that  $\text{SSy}_M(M') = \mathcal{X}$ . Then by Theorem 4.24, this  $M'$  has an end extension  $K$  which is  $\Sigma_1$ -closed for  $\text{I}\Delta_0 + \text{exp}$ .  $\square$

The previous corollary answers the question at the end of the Adamowicz–Cordón-Franco–Lara-Martín paper [7] about whether every countable Scott set is realized as the standard system of some structure  $\Sigma_1$ -closed for  $\text{I}\Delta_0 + \text{exp}$ . Alternatively, one can obtain an answer to their question directly, as the next theorem shows.

**Theorem 4.26.** For every countable  $(\mathbb{N}, \mathcal{X}) \models \text{WKL}_0^*$  and every  $\Pi_2$ -axiomatized consistent theory  $T \in \mathcal{X}$  extending  $\text{I}\Delta_0$ , there exists a structure  $K$  which is  $\Sigma_1$ -closed for  $T$  such that  $\text{SSy}(K) = \mathcal{X}$ .

*Proof.* Notice that consistency in  $(\mathbb{N}, \mathcal{X})$  is the same as real-world consistency. So we do not distinguish between the two. Let  $(D_n)_{n \in \mathbb{N}}$  be an enumeration of  $\mathcal{X}$ . We will iteratively find

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$

in  $\mathcal{X}$ , each of which will be consistent, complete, and Henkinized. The Henkin models of  $T_0, T_1, T_2, \dots$  will be called  $K_0, K_1, K_2, \dots$  respectively. The theory  $T_0$  can be any extension of  $T$  given by Corollary 3.3 that satisfies the inductive conditions. For each  $n \in \mathbb{N}$ , let  $\mathcal{L}_n$  be the language of  $T_n$ . Fix an enumeration  $(\sigma_n)_{n \in \mathbb{N}}$  of all sentences in the language  $\bigcup_{n \in \mathbb{N}} \mathcal{L}_n$  with  $\Sigma_1$  shapes. We assume every such sentence appears infinitely often in the enumeration.

Suppose  $T_n$  is found. Denote by  $\Delta_0 \cap T_n$  the set of all elements of  $T_n$  with  $\Delta_0$  shapes. Define

$$\tau_n = \begin{cases} \sigma_n, & \text{if } \sigma_n \in \mathcal{L}_n \text{ and } T \cup (\Delta_0 \cap T_n) \cup \{\sigma_n\} \text{ is consistent;} \\ \top, & \text{otherwise.} \end{cases}$$

Here  $\top$  is the symbol for verum. Consider the theory

$$\begin{aligned} T_n^+ &= T \cup (\Delta_0 \cap T_n) \cup \{\tau_n\} \\ &\cup \{\check{m} \in_{\text{Ack}} d_n : m \in D_n\} \\ &\cup \{\check{m} \notin_{\text{Ack}} d_n : m \in M \setminus D_n\}, \end{aligned}$$

where  $d_n$  is a constant symbol new to  $\mathcal{L}_n$ . As in the proof of Proposition 4.4, we see that  $(\mathbb{N}, \mathcal{X}) \models \text{Con}(T_n^+)$ . Our theory  $T_{n+1}$  can be any consistent complete Henkinized extension of  $T_n^+$  in  $\mathcal{X}$  given by Corollary 3.3.

One can then verify that  $K = \bigcup_{n \in \mathbb{N}} K_n$  has the desired properties.  $\square$

Notice the theory of the  $\Sigma_1$ -closed structure constructed above need not be in  $\mathcal{X}$  because such theories must compute  $0'$  for many choices of  $T$ ; see Theorem 3 in [2] and Theorem 5.1 in [7].

All these bring us back to the question when a model of  $\text{B}\Sigma_1 + \text{exp}$  has a proper end extension satisfying  $\text{I}\Delta_0 + \text{exp}$ . Theorem 4.20 tells us that every countable short  $\Pi_1$ -recursively saturated model of  $\text{B}\Sigma_1 + \text{exp}$  has such an extension. As shown by Adamowicz [1], this can alternatively be achieved using a version of the Arithmetized Completeness Theorem. Her argument also works for uncountable models of countable cofinality. There are, however, countable models of  $\text{B}\Sigma_1 + \text{exp}$  which cannot be properly end-extended to a model of  $\text{I}\Delta_0 + \text{exp}$ . Adamowicz [4] attributes this to Wilkie–Paris [74].

**Theorem 4.27.** There is a countable model  $M \models \text{B}\Sigma_1 + \text{exp}$  that has no proper end extension  $K \models \text{I}\Delta_0 + \text{exp}$ .

*Proof.* Let  $M_0$  be a structure  $\Sigma_1$ -closed for  $\text{I}\Delta_0 + \text{exp}$ . Follow the argument in Wilkie–Paris [74, pages 158–160] to find a countable  $M \models \text{B}\Sigma_1 + \text{exp}$  such that  $\Sigma_1\text{-Th}(M) = \Sigma_1\text{-Th}(M_0)$  and  $\Sigma_1\text{-Th}(M) \notin \text{SSy}(M)$ .

Suppose  $K \supseteq_e M$  satisfying  $\text{I}\Delta_0 + \text{exp}$ . Take  $b \in K \setminus M$ . Notice  $\Sigma_1\text{-Th}(K) = \Sigma_1\text{-Th}(M)$  because  $M_0$  is  $\Sigma_1$ -closed. Therefore, by Proposition 2.1,

$$\Sigma_1\text{-Th}(M) = \{\exists \bar{x} \xi(\bar{x}) : \xi \in \Delta_0 \text{ and } K \models \exists \bar{x} < b \xi(\bar{x})\} \in \text{SSy}(K) = \text{SSy}(M).$$

This is the required contradiction.  $\square$



Analogously, one may ask for which models  $(M, \mathcal{X}) \models \text{WKL}_0^*$  we can find  $K \models \text{I}\Delta_0 + \text{exp}$  properly end extending  $M$  such that  $\text{SSy}_M(K) = \mathcal{X}$ . By Theorem 4.6 and Corollary 4.10, we can find such an extension for every countably generated short  $\Pi_1^0$ -recursively saturated model  $(M, \mathcal{X}) \models \text{WKL}_0^*$ . As Theorem 5.4 in the next section shows, the same is also true for all countable models of  $\text{WKL}_0^*$  in which  $\mathbb{N}$  is not parameter-free  $\Delta_0(\Sigma_1)$ -definable.

## 5. A MENAGERIE OF INITIAL SEGMENTS

The focus of this section is on initial segments of models of  $\text{B}\Sigma_1 + \text{exp}$ . As one may expect, some results are dual to what we obtained in the previous section; compare Theorem 4.20 and Theorem 5.10, for example. Self-embeddings whose images are initial segments are self-dual in this sense. More importantly, they reveal a less obvious layer of connections between the previous section and the current one: having arbitrarily large initial segments satisfying a theory  $T$  is often equivalent to having an end extension to a model of  $T$  that preserves the  $\Pi_1$  theory.

Self-embeddings of countable models of first-order arithmetic have been well studied since Friedman [27]. The following is essentially Theorem 2.3 in Dimitracopoulos–Paris [22], albeit with a slightly weaker hypothesis. It improves Solovay’s self-embedding theorem [48, Theorem 4] for countable recursively saturated models of  $\text{B}\Sigma_1$ .

**Proposition 5.1.** Let  $M$  be a countable short  $\Pi_1$ -recursively saturated model of  $\text{B}\Sigma_1$ . If  $a, b \in M$  such that

$$M \models \exists x \xi(x, a) \rightarrow \exists x < b \xi(x, a)$$

for all  $\xi \in \Delta_0$ , then there is an embedding  $f: M \rightarrow M$  fixing  $a$  for which  $\text{Im}(f) \subseteq_e M$  below  $b$ .

*Proof.* Straightforward adaptation of the arguments in Chapter 12 of Kaye [35].  $\square$

Lessan [43, Section 5.1] was first to show that some saturation is necessary for a model of arithmetic to be isomorphic to a proper initial segment of itself. He called his saturation conditions *1-tallness* and *1-extendability*.

**Definition** (Lessan). Let  $a \in M \models \text{I}\Delta_0$ . We say  $M$  is *1-tall over  $a$*  if there is  $b \in M$  such that for all  $\xi \in \Delta_0$ ,

$$M \models \exists x \xi(x, a) \rightarrow \exists x < b \xi(x, a).$$

The model  $M$  is *1-extendable over  $a$*  if it has a proper end extension satisfying  $\Pi_1\text{-Th}(M, a)$ .

In other words, a model  $M \models \text{I}\Delta_0$  is 1-tall if and only if its  $\Delta_0$ -definable elements are not cofinal. The model  $M$  is 1-extendable if and only if it has a proper end extension to a model  $K \models \text{I}\Delta_0$  in which the  $\Delta_0$ -definable elements are not downward cofinal in  $K \setminus M$ . In fact, in *every* proper end extension  $K \models \text{I}\Delta_0$  of a sufficiently saturated model  $M \models \text{B}\Sigma_1 + \text{exp}$ , the  $\Delta_0$ -definable elements are not downward cofinal in  $K \setminus M$ ; see Theorem 5.3 below and Theorem 2.3 in Dimitracopoulos–Paris [22].

If  $a \in M \models \text{I}\Delta_0$  and  $M$  is 1-tall over  $a$ , then  $M$  has a proper initial segment  $I$  which contains  $a$  and is closed under multiplication such that  $\Pi_1\text{-Th}(I, a) = \Pi_1\text{-Th}(M, a)$ . Therefore, every 1-tall model of  $\text{I}\Delta_0$  has a 1-extendable proper initial segment. Conversely, every 1-extendable model of  $\text{I}\Delta_0$  has a 1-tall proper end extension. Thus if  $M \models \text{exp}$ , then one can code the  $\Sigma_1$  theory, or equivalently, the  $\Pi_1$  theory, of 1-tall or 1-extendable models as in the proof of Theorem 4.27.

**Lemma 5.2.** Let  $M$  be a nonstandard model of  $\text{I}\Delta_0 + \text{exp}$  and  $a \in M$ . If  $M$  is either 1-tall over  $a$  or 1-extendable over  $a$ , then  $\Pi_1\text{-Th}(M, a) \in \text{SSy}(M)$ .  $\square$

Dimitracopoulos and Paris [22] found a way to formulate Lessan's saturation notions in terms of the undefinability of the standard cut.

**Definition.** If  $n \in \mathbb{N}$ , then we denote by  $\Delta_0(\Sigma_n)$  the closure of  $\Sigma_n$  under Boolean operations and bounded quantification. Let  $M$  be an  $\mathcal{L}_A$  structure. For  $a \in M$ , say a subset of  $M$  is  $\Delta_0^-(\Sigma_1)$ -definable over  $a$  in  $M$  if it is definable over  $a$  by a  $\Delta_0(\Sigma_1)$  formula in which the parameter  $a$  appears only in the scope of an unbounded quantifier. A subset of  $M$  is  $\Delta_0^-(\Sigma_1)$ -definable in  $M$  if it is  $\Delta_0^-(\Sigma_1)$ -definable over some  $a \in M$ .

Evidently, parameter-free  $\Delta_0(\Sigma_1)$  definability is the same as  $\Delta_0^-(\Sigma_1)$  definability over 0. Notice also that both  $\Sigma_1$  definability and  $\Pi_1$  definability imply  $\Delta_0^-(\Sigma_1)$  definability. So if the standard cut  $\mathbb{N}$  is not  $\Delta_0^-(\Sigma_1)$ -definable in a model  $M \models \text{B}\Sigma_1 + \text{exp}$ , then  $M$  is both  $\Sigma_1$ - and short  $\Pi_1$ -recursively saturated.

**Theorem 5.3** (Lessan, Dimitracopoulos–Paris). Let  $a \in M \models \text{B}\Sigma_1 + \text{exp}$ , where  $M$  is countable. The following are equivalent.

- (a) There is an embedding  $f: M \rightarrow M$  fixing  $a$  for which  $\text{Im}(f) \subsetneq_e M$ .
- (b)  $\mathbb{N}$  is not  $\Delta_0^-(\Sigma_1)$ -definable in  $M$  over  $a$ .
- (c) No unbounded subset of  $\mathbb{N}$  is  $\Delta_0^-(\Sigma_1)$ -definable in  $M$  over  $a$ .
- (d) No unbounded subset of  $\mathbb{N}$  is definable in  $M$  by a formula of the form  $\varphi(v, a) \wedge \neg\psi(v, a)$  where  $\varphi, \psi \in \Sigma_1$ .
- (e) For no  $\varphi, \psi \in \Sigma_1$  is  $\mathbb{N} = \{v \in M : M \models \exists x < v (\varphi(v, x, a) \wedge \neg\psi(v, x, a))\}$ .
- (f) There is a nonstandard  $\nu \in M$  such that  $\{x \in M : M \models \exists \theta < \nu (\Delta_0\text{-Sat}(\theta(\check{x}, \check{a})) \wedge \forall y < x \neg\Delta_0\text{-Sat}(\theta(\check{y}, \check{a})))\} \not\subseteq_{\text{cf}} M$ .
- (g)  $M$  is 1-tall over  $a$ , and for every  $K \supseteq_e M$  satisfying  $\text{I}\Delta_0$ , there is  $K'$  such that  $M \subsetneq_e K' \subsetneq_e K$  and  $K' \models \Pi_1\text{-Th}(M, a)$ .
- (h)  $M$  is 1-tall over  $a$  and 1-extendable over  $a$ .

*Proof.* The implication (a)  $\Rightarrow$  (b) is the same as (i)  $\Rightarrow$  (vi) in Theorem 2.2 of [22]. For (b)  $\Rightarrow$  (c), if  $\varphi \in \Delta_0(\Sigma_1)$  such that  $\{v \in M : M \models \varphi(v, a)\} \subseteq_{\text{cf}} \mathbb{N}$ , then

$$\mathbb{N} = \{b \in M : M \models \exists v < b \varphi(v, a) \rightarrow \exists v < b (\varphi(v, a) \wedge \forall u < b (\varphi(u, a) \rightarrow u \leq v))\}.$$

The implication (c)  $\Rightarrow$  (d) is clear. One can show (d)  $\Rightarrow$  (e) using a pairing function. The proofs of (e)  $\Rightarrow$  (f) and (f)  $\Rightarrow$  (g) can be found in that of (iii)  $\Rightarrow$  (ii) in Theorem 2.2 of [22]. Trivially, we have (g)  $\Rightarrow$  (h). The implication (h)  $\Rightarrow$  (a) follows from Proposition 5.1.4 in Lessan [43]; see also Theorem 2.1 in [22]. Let us demonstrate how this last implication can be proved using what we have established in this paper. The essence of our proof seems only mildly different from that of Lessan's. This is the only part where the countability assumption on  $M$  cannot be weakened to  $\text{WKL}_0^*$ -expandability.

Suppose  $M$  is 1-tall over  $a$  and 1-extendable over  $a$ . Let  $b \in M$  such that

$$M \models \exists x \xi(x, a) \rightarrow \exists x < b \xi(x, a)$$

for all  $\xi \in \Delta_0$ . Let  $K \supseteq_e M$  satisfying  $\Pi_1\text{-Th}(M, a)$ . Then  $K$  satisfies the formula displayed above too. By Corollary 2.10, we may assume  $K$  satisfies  $\text{B}\Sigma_1$  and is countable and recursively saturated. Proposition 5.1 then provides us with an embedding  $f: K \rightarrow K$  fixing  $a$  for which  $\text{Im}(f) \subseteq_e K$  below  $b$ . In particular,  $\text{Im}(f) \subseteq_e M$ . So  $f \upharpoonright M$  is what we want.  $\square$

The same amount of recursive saturation is actually sufficient for a second-order self-embedding theorem in the sense of Tanaka [68]. The possibility of having such

a theorem for  $\text{WKL}_0^*$  was first noted in some private correspondence between Keita Yokoyama and the second author in 2012, as reported in [39, page 229]. The self-embeddings can be built directly by modifying Tanaka's original back-and-forth argument. We will, however, use an adaptation of our proof of Theorem 5.3 as in one of the first author's previous papers [26].

**Theorem 5.4.** Let  $(M, \mathcal{X})$  be a countable model of  $\text{WKL}_0^*$  and  $a, b \in M$ . If  $\mathbb{N}$  is not  $\Delta_0^-(\Sigma_1)$ -definable over  $a$  in  $M$  and for all  $\xi \in \Delta_0$ ,

$$M \models \exists x \xi(x, a) \rightarrow \exists x < b \xi(x, a),$$

then there is an embedding  $f: M \rightarrow M$  fixing  $a$  such that  $\text{Im}(f) \subseteq_e M$  below  $b$ , and

$$\text{SSy}_{\text{Im}(f)}(M) = \{f(X) : X \in \mathcal{X}\}.$$

*Proof.* Essentially the same as that of (h)  $\Rightarrow$  (a) for Theorem 5.3, using Theorem 4.6 and Theorem 5.3(g) in place of 1-extendability.  $\square$

As is well known [17, Corollary 6], no nonstandard model of  $\text{I}\Sigma_1$  can  $\Delta_0(\Sigma_1)$ -define  $\mathbb{N}$ . Therefore, Theorem 5.4 implies that every countable nonstandard model of  $\text{WKL}_0$  is isomorphic to a proper initial segment of itself. This is a particular case of Tanaka's self-embedding theorem.

One can view Corollary 4.15 as a preservation theorem for  $\Pi_1$  formulas: it essentially says that the  $\Pi_1$  formulas are exactly those preserved under taking initial segments closed under exponentiation in models of PA. There is also an analogue for  $\Pi_2$  formulas, i.e., one which essentially says that the  $\Pi_2$  formulas are exactly those preserved under taking unions of end chains of models of PA.

**Theorem 5.5.** Let  $M$  be a  $\text{WKL}_0^*$ -expandable model of  $\text{B}\Sigma_1 + \text{exp}$  in which  $\mathbb{N}$  is not  $\Delta_0^-(\Sigma_1)$ -definable. Let  $a \in M$  and  $\Phi(z)$  be a complete set of  $\mathcal{L}_A$  formulas extending PA. The following are equivalent.

- (a)  $\Pi_n \cap \Phi \in \text{SSy}(M)$  for every  $n \in \mathbb{N}$ , and  $M \models \Pi_2\text{-Th}(\Phi(a))$ .
- (b) For every  $a' \in M$ , there exists  $K \models \Pi_1\text{-Th}(M, a') + \Phi(a)$  end extending  $M$ .
- (c) There are arbitrarily large proper initial segments of  $M$  satisfying  $\Phi(a)$ .

*Proof.* Using the completeness of  $\Phi$ , it is straightforward to verify the implication (c)  $\Rightarrow$  (a). So we concentrate on (a)  $\Rightarrow$  (b) and (b)  $\Rightarrow$  (c).

Suppose (a) holds. Expand  $M$  to  $(M, \mathcal{X}) \models \text{WKL}_0^*$ . Notice  $\Pi_1\text{-Th}(M, a') \in \text{SSy}(M)$  by Theorem 5.3 and Lemma 5.2. To show (b), it suffices to find consistent complete Henkinized theories  $T_0, T_1, T_2, \dots$  in  $(M, \mathcal{X})$  such that for every  $n \in \mathbb{N}$ ,

- the language of  $T_n$  consists of the symbols in  $\mathcal{L}_A^M$  and some extra constant symbols;
- $T_n \supseteq \Pi_1\text{-Th}(M, \check{a}') \cup (\Pi_{n+2} \cap \Phi(\check{a}))$ ; and
- $T_{n+1} \supseteq \Pi_{n+1}^M \cap T_n$ , where  $\Pi_{n+1}^M \cap T_n$  denotes the set of all formulas from  $T_n$  with  $\Pi_{n+1}^M$  shapes.

Finding  $T_1, T_2, \dots$  is the same as in our proof of Theorem 4.14. For  $T_0$ , in view of short  $\Pi_1$  recursive saturation (and the trick involving conjunctions in the proof of Theorem 4.14), we only need to show  $M \models \text{Con}(\sigma(\check{a}') \wedge \varphi(\check{a}))$  whenever  $\sigma(a') \in \Pi_1\text{-Th}(M, a')$  and  $\varphi \in \Pi_2 \cap \Phi$ . Fix such  $\sigma(a')$  and  $\varphi$ . Take any  $K'_0 \succ_{\Pi_1} M$  satisfying  $\Phi(a)$ . By elementarity, we know  $K'_0 \models \sigma(a') \wedge \varphi(a)$ . Uniform reflection in  $K'_0$ , as given by Theorem 4.11, then implies  $K'_0 \models \text{Con}(\sigma(\check{a}') \wedge \varphi(\check{a}))$ . So the same is true in  $M$ , as required.

Now assume (b) holds. Let  $a' \in M$  be arbitrary. We know  $M$  is 1-tall over  $\langle a, a' \rangle$  by Theorem 5.3. Find  $b \in M$  such that for all  $\xi \in \Delta_0$ ,

$$M \models \exists x \xi(x, a, a') \rightarrow \exists x < b \xi(x, a, a').$$

Apply (b) to get  $K \models \Pi_1\text{-Th}(M, a, a') + \Phi(a)$  end extending  $M$ . Let  $K_0$  be a countable elementary substructure of  $K$  containing  $a, a'$  and  $b$ . Notice  $K_0$  satisfies PA and is thus short  $\Pi_1$ -recursively saturated. So Proposition 5.1 gives us an embedding  $f: K_0 \rightarrow K$  fixing  $a$  and  $a'$  for which  $\text{Im}(f) \subseteq_e K_0$  below  $b$ . Then  $\sup_M \text{Im}(f) \supseteq \text{Im}(f) \models \Phi(a)$  by Gaifman's theorem [28, Theorem 3], i.e., Theorem 7.7 in [35].  $\square$

The equivalence of (a) and (c) above for  $M \models \text{PA}$  is essentially Lemma 5.2.2(v) in Lessan [43]; see also Remark (ii) on page 58 of his thesis. As in Section 4, the completeness of  $\Phi$  is needed only for (c)  $\Rightarrow$  (a), and we can require recursive saturation exactly when  $\Phi \in \text{SSy}(M)$ ; cf. Lemma 5.2.2(iv) in Lessan [43].

**Theorem 5.6.** Let  $M$  be a  $\text{WKL}_0^*$ -expandable model of  $\text{B}\Sigma_1 + \text{exp}$  in which  $\mathbb{N}$  is not  $\Delta_0^-(\Sigma_1)$ -definable. Let  $a \in M$  and  $\Phi(z)$  be a complete set of  $\mathcal{L}_A$  formulas extending PA. The following are equivalent.

- (a)  $\Phi \in \text{SSy}(M)$  and  $M \models \Pi_2\text{-Th}(\Phi(a))$ .
- (b) For every  $a' \in M$ , there exists a recursively saturated  $K \models \Pi_1\text{-Th}(M, a') + \Phi(a)$  end extending  $M$ .
- (c) There are arbitrarily large recursively saturated proper initial segments of  $M$  satisfying  $\Phi(a)$ .

*Proof.* Given the fact that cofinal extension preserves recursive saturation [65], this is a straightforward adaptation of the proof of Theorem 5.5.  $\square$

Under extra countability or codedness assumptions, the hypotheses in Theorem 5.5 and Theorem 5.6 can be significantly weakened using the method of indicators. The most delicate part is in identifying the conditions from which one can obtain a sufficiently large interval to start the construction. The rest is routine verification.

**Lemma 5.7.** Let  $a \in M \models \text{I}\Delta_0$ , where  $M$  is  $\Sigma_1$ -recursively saturated. Let  $\mathcal{L}_A^*$  be a recursive language extending  $\mathcal{L}_A$  and  $\Phi(z)$  be a set of  $\mathcal{L}_A^*$  formulas extending  $\text{B}\Sigma_1$  in  $\text{SSy}(M)$ . The following are equivalent.

- (a)  $M \models \Pi_2\text{-Th}(\Phi(a))$ .
- (b) For every large enough  $a' \in M$ , there exists  $b \in M$  such that for all  $\zeta \in \Delta_0$ ,

$$\Phi(a) \vdash \forall x < a' \exists y \zeta(x, y, a, a') \quad \Rightarrow \quad M \models \forall x < a' \exists y < b \zeta(x, y, a, a').$$

*Proof.* Showing (b)  $\Rightarrow$  (a) is straightforward. So let us concentrate on the converse. Suppose (a) holds. Pick any  $a' \in M$  bigger than  $a$ . By Craig's Trick and  $\Delta_0$  recursive saturation, it suffices to show the finite satisfiability of

$$p(b) = \{\forall x < a' \exists y < b \zeta(x, y, a, a') : \zeta \in \Delta_0 \text{ and } \Phi(a) \vdash \forall x < a' \exists y \zeta(x, y, a, a')\}$$

in  $M$ . Let  $\zeta \in \Delta_0$  such that  $\Phi(a) \vdash \forall x < a' \exists y \zeta(x, y, a, a')$ . Then

$$\Phi(a) \vdash \forall w \exists b \forall x < w \exists y < b \zeta(x, y, a, w)$$

because  $\Phi(a) \supseteq \text{B}\Sigma_1$  and  $a'$  does not appear in  $\Phi(a)$ . So the same formula is true in  $M$  since  $M \models \Pi_2\text{-Th}(\Phi(a))$ . Thus  $M \models \exists b \forall x < a' \exists y < b \zeta(x, y, a, a')$ , as required.  $\square$

Recall that a model of  $\text{I}\Delta_0 + \text{exp}$  is  $\Sigma_1$ -recursively saturated if  $\mathbb{N}$  is not  $\Sigma_1$ -definable in it.

**Theorem 5.8** (essentially Kaye). Let  $M$  be a  $\Sigma_1$ -recursively saturated model of  $\text{I}\Delta_0 + \text{exp}$ . Let  $a \in M$  and  $\Phi(z)$  be a set of  $\mathcal{L}_A$  formulas extending  $\text{B}\Sigma_1$  in  $\text{SSy}(M)$ . The following are equivalent.

- (a)  $M \models \Pi_2\text{-Th}(\Phi(a))$ .

(c) There are arbitrarily large proper initial segments of  $M$  satisfying  $\Phi(a)$ .

*Proof.* Use Lemma 5.7 with the arguments in Section 14.2 of Kaye's book [35].  $\square$

As Theorem 5.5 and Theorem 5.6 suggest, the existence of arbitrarily large initial segments satisfying a theory is often equivalent to the existence of end extensions satisfying the same theory in which a certain  $\Pi_1$  theory is preserved. We can fill in this missing clause for Theorem 5.8 under a different set of hypotheses.

**Corollary 5.9.** Let  $M$  be a short  $\Pi_1$ -recursively saturated countable nonstandard model of  $\text{BS}\Sigma_1$  in which  $\Pi_1\text{-Th}(M, a') \in \text{SSy}(M)$  for every  $a' \in M$ . Let  $\mathcal{L}_A^*$  be a recursive language extending  $\mathcal{L}_A$  and  $\Phi(z)$  be a set of  $\mathcal{L}_A^*$  formulas in  $\text{SSy}(M)$ . The following are equivalent for all  $a \in M$ .

- (a)  $M \models \Pi_2\text{-Th}(\Phi(a))$ .
- (b) For every  $a' \in M$ , there exists an end extension of  $M$  that expands to a (recursively saturated) model of  $\Pi_1\text{-Th}(M, a') + \Phi(a)$ .

*Proof.* First, suppose (a) holds. Take any  $a' \in M$ . Without loss of generality, assume  $a' \neq a$ . In view of Theorem 4.20, it suffices to show  $M \models \Pi_1\text{-Th}(\Pi_1\text{-Th}(M, a, a') + \Phi(a))$ . Take  $\xi, \zeta \in \Pi_1$  such that

$$M \models \xi(a, a') \quad \text{and} \quad \Phi(a) \vdash \xi(a, a') \rightarrow \zeta(a, a').$$

Since  $a'$  does not appear in  $\Phi(a)$ , we know  $\Phi(a) \vdash \forall w (\xi(a, w) \rightarrow \zeta(a, w))$ . So this  $\Pi_2$  formula is true in  $M$  by (a). It follows that  $M \models \zeta(a, a')$ , as required.

Conversely, suppose (b) holds. Let  $\eta \in \Delta_0$  such that  $\Phi(a) \vdash \forall x \exists y \eta(x, y, a)$ . We want to show  $M \models \forall x \exists y \eta(x, y, a)$ . Take any  $a' \in M$ . Using (b), find  $K \supseteq_e M$  that expands to a model of  $\Pi_1\text{-Th}(M, a, a') + \Phi(a)$ . Our choice of  $\eta$  implies  $K \models \exists y \eta(a', y, a)$ . So the same formula is true in  $M$  too.  $\square$

Kaye [35, page 204] observed that countability can help make the initial segments in Theorem 5.8 recursively saturated. We formulate this in the style of Theorem 4.20. Recall our convention that if  $b \in M \models \text{I}\Delta_0$ , then we use again the letter  $b$  to refer to  $\{x \in M : x < b\}$  enriched with the graphs of the arithmetic operations from  $M$ .

**Theorem 5.10** (essentially Kaye). Let  $a, b \in M \models \text{I}\Delta_0$  in which  $b$  is countable and recursively saturated. Let  $\mathcal{L}_A^*$  be a recursive language extending  $\mathcal{L}_A$  and  $\Phi(z)$  be a set of  $\mathcal{L}_A^*$  formulas extending  $\text{BS}\Sigma_1$  in  $\text{SSy}(M)$ . The following are equivalent.

- (a) For all formulas  $\xi \in \Delta_0$ ,
 
$$\Phi(a) \vdash \exists x \xi(x, a) \quad \Rightarrow \quad M \models \exists x < b \xi(x, a).$$
- (b) Some initial segment of  $M$  not containing  $b$  expands to a recursively saturated model of  $\Phi(a)$ .

*Proof.* One can use Lemma 5.7 and the arguments in Kaye [35, pages 201, 202] to show this. Here, we will unravel Kaye's argument into a direct Henkin construction similar to that in our proof of Theorem 4.20.

Suppose (a) holds. If  $X \subseteq M$ , then write  $\mathcal{L}_A^*(X)$  for the language obtained from  $\mathcal{L}_A^*$  by adding a new constant symbol for every element of  $X$ . We will find sets of  $\mathcal{L}_A^*(M)$  sentences

$$\Phi(a) = \Phi_0 \subseteq \Phi_1 \subseteq \Phi_2 \subseteq \dots$$

by recursion. During the construction, we assume inductively that for every  $n \in \mathbb{N}$ ,

- (i)  $\Phi_n \in \text{SSy}(M)$ ;
- (ii)  $\Phi_n$  involves only finitely many parameters  $\bar{a} \in M$ ; and

(iii) for all formulas  $\xi \in \Delta_0$ ,

$$\Phi_n(\bar{a}) \vdash \exists x \xi(x, \bar{a}) \quad \Rightarrow \quad M \models \exists x < b \xi(x, \bar{a}).$$

By our hypotheses, we know  $\Phi_0$  satisfies these conditions. The construction will be arranged such that the set of parameters from  $M$  mentioned in  $\Phi_\omega = \bigcup_{n \in \mathbb{N}} \Phi_n$  is an initial segment  $I \subseteq_e M$  below  $b$ , and such that  $\Phi_\omega$  is consistent, complete, and Henkinized as an  $\mathcal{L}_A^*(I)$  theory. This  $I$  is the initial segment we want.

Let us first make  $\Phi_\omega$  complete. Suppose  $\Phi_n$  is found. Let  $a_0, a_1, \dots, a_\ell$  be the parameters from  $M$  that appear in  $\Phi_n$  in increasing order. Consider a formula  $\theta \in \mathcal{L}_A^*$  and parameters  $c_1, c_2, \dots, c_k \in M$  below  $a_\ell$ . If setting  $\Phi_{n+1} = \Phi_n \cup \{\theta(\bar{c}, \bar{a})\}$  preserves the inductive conditions, then we are already done. So suppose not. Without loss of generality, assume the  $a_i$ 's are distinct from the  $c_j$ 's. Take  $\xi \in \Delta_0$  such that

$$\Phi_n(\bar{a}) \cup \{\theta(\bar{c}, \bar{a})\} \vdash \exists x \xi(\bar{c}, x, \bar{a}) \quad \text{and} \quad M \models \forall x < b \neg \xi(\bar{c}, x, \bar{a}).$$

It suffices to show  $\Phi_{n+1} = \Phi_n \cup \{-\theta(\bar{c}, \bar{a})\}$  again satisfies the inductive conditions. Let  $\zeta \in \Delta_0$  such that  $\Phi_n(\bar{a}) \cup \{-\theta(\bar{c}, \bar{a})\} \vdash \exists x \zeta(\bar{c}, x, \bar{a})$ . Then

$$\Phi_n(\bar{a}) \vdash \exists x (\xi(\bar{c}, x, \bar{a}) \vee \zeta(\bar{c}, x, \bar{a})).$$

Since the parameters  $\bar{c}$  do not appear in  $\Phi_n(\bar{a})$ ,

$$\begin{aligned} & \Phi_n(\bar{a}) \vdash \forall \bar{v} < a_\ell \exists x (\xi(\bar{v}, x, \bar{a}) \vee \zeta(\bar{v}, x, \bar{a})) \\ \therefore & \Phi_n(\bar{a}) \vdash \exists d \forall \bar{v} < a_\ell \exists x < d (\xi(\bar{v}, x, \bar{a}) \vee \zeta(\bar{v}, x, \bar{a})) && \text{since } \Phi \supseteq \text{BS}\Sigma_1, \\ \therefore & M \models \exists d < b \forall \bar{v} < a_\ell \exists x < d (\xi(\bar{v}, x, \bar{a}) \vee \zeta(\bar{v}, x, \bar{a})) && \text{by condition (iii),} \\ \therefore & M \models \exists x < b (\xi(\bar{c}, x, \bar{a}) \vee \zeta(\bar{c}, x, \bar{a})) && \text{since } \bar{c} < a_\ell. \end{aligned}$$

As  $M \models \forall x < b \neg \xi(\bar{c}, x, \bar{a})$ , we conclude that  $M \models \exists x < b \zeta(\bar{c}, x, \bar{a})$ , as required.

Next, we show how to ensure  $I$  is recursively saturated. This will finish the proof because it will Henkinize  $\Phi_\omega$  in particular. Suppose  $\Phi_n$  is found. Let  $a_0, a_1, \dots, a_\ell$  list all the parameters from  $M$  that appear in  $\Phi_n$ . Consider a recursive set of  $\mathcal{L}_A^*(M)$  formulas  $p(v, \bar{a})$  with only the parameters shown. For simplicity, assume that given any two formulas in  $p$ , one can always find a third one in  $p$  which logically implies the first two. If there is  $\psi(v, \bar{a}) \in p(v, \bar{a})$  such that  $\Phi_{n+1} = \Phi_n \cup \{\neg \exists v \psi(v, \bar{a})\}$  satisfies the inductive conditions, then we are already done. So suppose not. Given what we proved in the previous paragraph, we may (and will) assume, without loss of generality, that  $\Phi_n(\bar{a})$  contains  $\exists v \psi(v, \bar{a})$  for every  $\psi \in p$ . If there is  $i \leq \ell$  such that  $\Phi_{n+1} = \Phi_n \cup p(a_i, \bar{a})$  satisfies the inductive conditions, then we are also finished. So suppose this is also not the case. We thus get  $\zeta_0, \zeta_1, \dots, \zeta_\ell \in \Delta_0$  such that

$$\Phi_n(\bar{a}) \cup p(a_i, \bar{a}) \vdash \exists x \zeta_i(x, \bar{a}) \quad \text{and} \quad M \models \forall x < b \neg \zeta_i(x, \bar{a})$$

for all  $i \leq \ell$ . Consider

$$q(v) = \{\exists x < b \xi(v, x, \bar{a}) : \xi \in \Delta_0 \text{ and } \Phi_n(\bar{a}) \cup p(v, \bar{a}) \cup \{v \neq a_i : i \leq \ell\} \vdash \exists x \xi(v, x, \bar{a})\}.$$

By Craig's Trick, we see that  $q$  is logically equivalent to a set of  $\Delta_0$  formulas in  $\text{SSy}(M)$ . Let us show the finite satisfiability of  $q$  in  $M$ . Take  $\psi \in p$  and  $\xi \in \Delta_0$  such that

$$\Phi_n(\bar{a}) \cup \{\psi(v, \bar{a})\} \cup \{v \neq a_i : i \leq \ell\} \vdash \exists x \xi(v, x, \bar{a}).$$

Without loss of generality, assume  $\Phi_n(\bar{a}) \cup \{\psi(a_i, \bar{a})\} \vdash \exists x \zeta_i(x, \bar{a})$  for every  $i \leq \ell$ . Our earlier assumption that  $\Phi_n(\bar{a})$  contains  $\exists v \psi(v, \bar{a})$  then implies

$$\Phi_n(\bar{a}) \vdash \bigvee_{i \leq \ell} \exists x \zeta_i(x, \bar{a}) \vee \exists v \exists x \xi(v, x, \bar{a}).$$

So  $M \models \bigvee_{i \leq \ell} \exists x < b \zeta_i(x, \bar{a}) \vee \exists v < b \exists x < b \xi(v, x, \bar{a})$  by inductive condition (iii). Our choice of the  $\zeta_i$ 's then tells us  $M \models \exists v < b \exists x < b \xi(v, x, \bar{a})$ , as required.

Recall that  $b$  interprets  $b^2$  in a natural way. Via this natural interpretation, the  $\Delta_0$  type  $q(v)$  over  $M$  translates into a recursive type over the relational structure  $b$ . So we obtain  $m \in M$  realizing  $q$  by the recursive saturation of  $b$ . Notice  $m \notin \{a_0, a_1, \dots, a_\ell\}$  because  $q(v)$  contains a formula trivially equivalent to  $v \neq a_i$  for each  $i \leq \ell$ . Set  $\Phi_{n+1} = \Phi_n \cup p(m, \bar{a})$ . If  $\Phi_{n+1}(m, \bar{a}) \vdash \exists x \xi(m, x, \bar{a})$ , where  $\xi \in \Delta_0$ , then

$$\Phi_n(\bar{a}) \cup p(v, \bar{a}) \cup \{v \neq a_i : i \leq \ell\} \vdash \exists x \xi(v, x, \bar{a})$$

as  $m \notin \{a_0, a_1, \dots, a_\ell\}$ , and so  $\exists x < b \xi(v, x, \bar{a})$  is in  $q(v)$ , making  $M \models \exists x < b \xi(m, x, \bar{a})$ . Therefore, the inductive conditions are again satisfied.  $\square$

*Remark 5.11.* Theorem 5.10 generalizes naturally to every level of the arithmetic hierarchy: if  $n \in \mathbb{N}$ , then we can make the initial segment  $\Pi_n$ -elementary in (b) by

- requiring  $M$  to realize all recursive  $\Delta_0^+(\Pi_n)$  types  $q(v)$  which contain the formula  $v < b$  and involve only parameters below  $b$ ;
- requiring  $\Phi(z) \supseteq \text{B}\Sigma_{n+1}$ ; and
- replacing  $\Delta_0$  in (a) with  $\Delta_0^+(\Pi_n)$ ,

where  $\Delta_0^+(\Pi_n)$  denotes the closure of  $\Pi_n$  under bounded quantification (but, unlike the case in the definition of  $\Delta_0(\Sigma_n)$ , not under negation). As in Remark 4.21, if  $\Phi$  is recursive, then one can weaken  $\text{I}\Delta_0$  and  $\text{B}\Sigma_{n+1}$  respectively to  $\text{PA}^-$  and the induction-free  $\Sigma_{n+1}$  collection scheme here.

Recall that if  $M$  is a short  $\Delta_0$ -recursively saturated model of  $\text{I}\Delta_0$ , then every  $b \in M$  is recursively saturated as a substructure of  $M$ .

**Corollary 5.12.** Let  $a \in M \models \text{I}\Delta_0$ , where  $M$  is countable and  $\Sigma_1$ -recursively saturated. Let  $\mathcal{L}_A^*$  be a recursive language extending  $\mathcal{L}_A$  and  $\Phi(z)$  be a set of  $\mathcal{L}_A^*$  formulas extending  $\text{B}\Sigma_1$  in  $\text{SSy}(M)$ . The following are equivalent.

- (a)  $M \models \Pi_2\text{-Th}(\Phi(a))$ .
- (c) There are arbitrarily large proper initial segments of  $M$  that expand to (recursively saturated) models of  $\Phi(a)$ .

*Proof.* This follows directly from Lemma 5.7 and Theorem 5.10.  $\square$

We also have an analogous preservation theorem for  $\Sigma_1$  formulas; cf. Guaspari [30, Theorem 6.5].

**Corollary 5.13.** Let  $a \in M \models \text{I}\Delta_0$ , where  $M$  is countable and  $\Sigma_1$ -recursively saturated. Let  $\mathcal{L}_A^*$  be a recursive language extending  $\mathcal{L}_A$  and  $\Phi(z)$  be a set of  $\mathcal{L}_A^*$  formulas extending  $\text{B}\Sigma_1$  in  $\text{SSy}(M)$ . The following are equivalent.

- (a)  $M \models \Sigma_1\text{-Th}(\Phi(a))$ .
- (b) Some proper initial segment of  $M$  expands to a (recursively saturated) model of  $\Phi(a)$ .

*Proof.* Use  $\Delta_0$  recursive saturation to find  $b \in M$  satisfying

$$\Phi(a) \vdash \exists x \xi(x, a) \quad \Rightarrow \quad M \models \exists x < b \xi(x, a)$$

for all  $\xi \in \Delta_0$ . Then apply Theorem 5.10.  $\square$

Corollary 5.13 characterizes  $\Sigma_1$  recursive saturation. In addition to being the dual of Corollary 4.22, one can view this as the level-one case of a hierarchical version of the fact that a model of  $\text{PA}$  is recursively saturated if and only if it has arbitrarily large recursively saturated elementary initial segments.

**Corollary 5.14.** The following are equivalent for a countable  $M \models \text{I}\Delta_0$ .

- (a)  $M$  is  $\Sigma_1$ -recursively saturated.
- (b) Whenever  $a \in M$  and  $\Phi(z)$  is a recursive set of formulas extending  $\text{B}\Sigma_1$  in a finite language extending  $\mathcal{L}_A$ , if  $M \models \Pi_2\text{-Th}(\Phi(a))$ , then there is  $I \subseteq_e M$  that expands to a model of  $\Phi(a)$ .

*Proof.* We have already seen one direction. For the converse, suppose (b) holds. Consider the recursive  $\Sigma_1$  type  $p(v, a)$  over  $M$  where  $a \in M$ . Let  $c$  be a constant symbol new to  $\mathcal{L}_A$ . Recall the  $\Pi_2$  conservativity of  $\text{B}\Sigma_1$  over  $\text{I}\Delta_0$  from Remark 3.8. If  $\varphi \in p$  and  $\xi \in \Pi_2$  such that  $\text{B}\Sigma_1 \cup \{\varphi(c, a)\} \vdash \xi(a)$ , then  $\text{I}\Delta_0 \vdash \exists v \varphi(v, a) \rightarrow \xi(a)$  by this conservativity result, and so  $M \models \xi(a)$  because  $p(v, a)$  is finitely satisfiable in  $M$ . Hence  $M \models \Pi_2\text{-Th}(\text{B}\Sigma_1 \cup p(c, a))$ . Apply (b) to  $\Phi(a) = \text{B}\Sigma_1 \cup p(c, a)$  to get  $I \subseteq_e M$  in which  $p$  is realized. Any realization of  $p$  in  $I$  realizes  $p$  in  $M$  because  $I \subseteq_e M$  and  $p(v, z) \subseteq \Sigma_1$ .  $\square$

## 6. QUESTIONS

Paris [48, Problem 1] asked whether every countable model of  $\text{B}\Sigma_1$  has a proper end extension to a model of  $\text{I}\Delta_0$ . As we saw in Proposition 4.1 and Theorem 4.20, if  $M$  is a countable model of  $\text{B}\Sigma_1$  that either satisfies  $\text{exp}$  or is short  $\Pi_1$ -recursively saturated, then it has a proper end extension  $K \models \text{I}\Delta_0$ . These are special cases of the results in the Wilkie–Paris paper [74]. Wilkie and Paris posed in the same paper the problem of finding a ‘natural’ necessary and sufficient condition under which a countable model of  $\text{B}\Sigma_1$  possesses a proper end extension satisfying  $\text{I}\Delta_0$ . There has been considerable effort in understanding this problem [1, 2, 3, 4, 18, 20, 23, 38, 74], but it is apparently difficult to find a satisfactory solution due to complexity theoretic issues. Therefore, it seems reasonable to consider related questions whose answers may be easier to obtain (and may shed some light on the main problem). For instance, it is not known whether Theorem 2.7 generalizes to all  $M \models \text{B}\Sigma_1 + \text{exp}$ .

**Question 6.1** (Dimitracopoulos–Paschalis). Does every model of  $\text{B}\Sigma_1 + \text{exp}$  have a proper end extension satisfying  $\text{I}\Delta_0$ ?

Dimitracopoulos and Paschalis listed this question and a few variations in their paper [23]. The following is perhaps most likely to have a positive answer.

**Question 6.2.** Does every short  $\Pi_1$ -recursively saturated model of  $\text{B}\Sigma_1 + \Omega_1$  of countable cofinality have a proper end extension satisfying  $\text{I}\Delta_0$ ?

As remarked at the end of Section 2, every  $(M, \mathcal{X}) \models \text{RCA}_0$  extends to  $(M, \mathcal{Y}) \models \text{WKL}_0$ . Apart from this exceptional case, it is unknown to the authors whether Theorem 2.11 and Theorem 2.12 generalize to uncountable models, even when the models are assumed to be countably  $\Delta_1^0$ -generated.

**Question 6.3.** Does every countably  $\Delta_1^0$ -generated model  $(M, \mathcal{X}) \models \text{RCA}_0^*$  have an extension  $(M, \mathcal{Y}) \models \text{WKL}_0^*$ ?

In view of the work by Adamowicz [1] described in Section 2, a positive answer to the following would help, at least when the first-order part of the ground model is of countable cofinality.

**Question 6.4.** Does every countably  $\Delta_1^0$ -generated model  $(M, \mathcal{X}) \models \text{RCA}_0^*$  have a finitely  $\Delta_1^0$ -generated extension  $(M, \mathcal{Y}) \models \text{RCA}_0^*$ ?

The analogous questions for countable models of  $\text{RCA}_0$  and  $\text{ACA}_0$  both have positive answers.

**Theorem 6.5** (Simpson–Tanaka–Yamazaki). Every countable  $(M, \mathcal{X}) \models \text{RCA}_0$  has a finitely  $\Delta_1^0$ -generated extension  $(M, \mathcal{Y}) \models \text{RCA}_0$ .



*Proof.* This is Theorem 5.11 in [61]. See Belanger [11] for a more direct argument.  $\square$

**Theorem 6.6** (Kossak–Schmerl). Every countable  $(M, \mathcal{X}) \models \text{ACA}_0$  has a finitely arithmetically generated extension  $(M, \mathcal{Y}) \models \text{ACA}_0$ .

*Proof.* This is Theorem 6.5.6 in [41].  $\square$

The first author proved in a previous paper [25, Theorem A] that Theorem 4.12 no longer holds if the ‘countably generated’ condition is changed to ‘size- $\aleph_1$ ’, even when  $M = \mathbb{N}$  and  $(M, \mathcal{X}) \models \text{ACA}_0$ . On the contrary, Knight and Nadel [37] observed that Theorem 4.6 remains true when one changes ‘countably generated’ to ‘size- $\aleph_1$ ’ provided  $M = \mathbb{N}$ . It is not clear whether the same holds without the proviso  $M = \mathbb{N}$ .

**Question 6.7.** Can one use an argument similar to our proof of Theorem 4.6 to show that for every  $(M, \mathcal{X}) \models \text{WKL}_0^*$  of size  $\aleph_1$ , there exists a proper end extension  $K \supset_e M$  in which  $\text{SSy}_M(K) = \mathcal{X}$ ?

It is not hard to construct a countable short  $\Pi_1$ -recursively saturated 1-extendable model of  $\text{B}\Sigma_1 + \text{exp}$  which is not 1-tall [22, page 17, remark (a)]. On the contrary, Dimitracopoulos and Paris [22, Theorem 2.3] showed that all countable short  $\Pi_1$ -recursively saturated 1-tall models of  $\text{B}\Sigma_1 + \text{exp}$  must be 1-extendable in a very strong sense. It is not clear whether the short  $\Pi_1$  recursive saturation condition here (or even in Proposition 5.1) can be omitted. The example of a 1-tall but not 1-extendable model that Lessan gave [43, Remarks after Definition 5.1.2] does not satisfy  $\text{B}\Sigma_1$ .

**Question 6.8.** Is there a countable 1-tall but not 1-extendable model of  $\text{B}\Sigma_1 + \text{exp}$ ?

It seems interesting to investigate the interpretability version of the Clote–Hájek–Paris question on the provability of the Friedman–Paris conservation result in  $\text{I}\Delta_0 + \text{exp}$  mentioned at the end of Section 3.

**Question 6.9.** Let  $n \in \mathbb{N}$ . Can one always find  $\sigma \in \Sigma_{n+2}$  such that  $\text{B}\Sigma_{n+1} + \text{exp} + \sigma$  is not interpretable in  $\text{I}\Sigma_n + \text{exp} + \sigma$ ?

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