Number of conjugacy classes of indicated cuts

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Recently, Kaye [1] discovered a notion of *generic cuts* in countable arithmetically saturated models of PA. These generic cuts have nice conjugacy properties: whenever two of them are sufficiently close to each other, they are *conjugate*, i.e., there is an automorphism of M mapping one onto the other. A natural question to ask is how many conjugacy classes of generic cuts there are. This note aims to answer this question. Along the way, some more general results on families of indicated cuts will be proved.

We assume some familiarity with Kaye's paper [1], particularly the definitions of indicators, generic cuts and related notions (e.g., Z_Y^M, *Y-intervals*, $P_Y(I)$, $M_Y(x)$, etc.). Throughout this paper, M is a countable model of PA, and Y is an indicator. To avoid triviality, we require our indicator Y to be *non-degenerate* in M, i.e.,

$$\forall n \in \mathbb{N}\ \ M \models \exists x \exists y\, Y(x,y) \geqslant n,$$

but absoluteness is not needed. $\mathscr{D}$ is a dense subset of Z_Y^M, and cuts in $\mathscr{D}$ are called *$\mathscr{D}$-cuts*. $(Y_n)_{n\in M}$ is the hierarchy of functions from M to M defined by

$$\forall n,x \in M\ \ Y_n(x) = \begin{cases} (\mu y)(Y(x,y) \geqslant n), & \text{if } M \models \exists y (Y(x,y) \geqslant n); \\ 0, & \text{otherwise.} \end{cases}$$

For $n,x \in M$, Y_n is said to be *properly defined* at x if and only if

$$M \models Y(x, Y_n(x)) \geqslant n.$$

$\mathrm{cl}(\varnothing)$ will denote the Skolem hull of M, and $\overline{\mathrm{cl}}(\varnothing)$ will denote its convex closure.

An obvious necessary condition for two cuts to be conjugate is the following.

Lemma 1. If two cuts in M are conjugate, then there is no definable point between them. □

Although this condition is not sufficient (cf. Theorem 6), Lemma 1 is essentially enough to show that there are many non-conjugate $\mathscr{D}$-cuts when $M \not\models \mathrm{Th}(\mathbb{N})$. The following observations will be useful.

Lemma 2 (Kaye [1, Proposition 2.4])**.** For all $x,y \in \mathrm{cl}(\varnothing)$ with $Y(x,y) > \mathbb{N}$, there exists $z \in \mathrm{cl}(\varnothing)$ such that $Y(x,z) > \mathbb{N}$ and $Y(z,y) > \mathbb{N}$. □

Lemma 3. Suppose $M \not\models \mathrm{Th}(\mathbb{N})$. If $a \in \mathrm{cl}(\varnothing)$ such that Y_n is properly defined at a for all $n \in \mathbb{N}$, then $M_Y(a) \subsetneq_{\mathrm{e}} \overline{\mathrm{cl}}(\varnothing)$.

Proof. Let $a \in \mathrm{cl}(\varnothing)$ such that Y_n is properly defined at a for all $n \in \mathbb{N}$. Then $(Y_n(a))_{n\in\mathbb{N}}$ is a sequence in $\mathrm{cl}(\varnothing)$ whose limit is $M_Y(a)$. So, $M_Y(a) \subseteq_{\mathrm{e}} \overline{\mathrm{cl}}(\varnothing)$.

Suppose $M_Y(a) = \overline{\mathrm{cl}}(\varnothing)$. Then since $\overline{\mathrm{cl}}(\varnothing) \prec M$, we have

$$n \in \mathbb{N} \quad \text{iff} \quad \overline{\mathrm{cl}}(\varnothing) \models \exists y Y(a,y) \geqslant n$$

for all $n \in \overline{\mathrm{cl}}(\varnothing)$, contradicting Σ_1-induction. $\square$

Here is the promised result.

Theorem 4. If $M \not\models \mathrm{Th}(\mathbb{N})$, then there are at least countably infinitely many conjugacy classes of $\mathscr{D}$-cuts that are contained in $\overline{\mathrm{cl}}(\varnothing)$.

Proof. By the non-degeneracy of Y, Y_n is properly defined at 0 for all $n \in \mathbb{N}$, and so $M_Y(0) \subsetneq_{\mathrm{e}} \overline{\mathrm{cl}}(\varnothing)$ by Lemma 3. Take $a \in \mathrm{cl}(\varnothing)$ such that $a > M_Y(0)$. Then $Y(0,a) > \mathbb{N}$. Using Lemma 2, one can divide the Y-interval $[0,a]$ indefinitely into smaller subintervals by definable points. Since $\mathscr{D}$ is dense in Z_Y^M, we get countably infinitely many mutually non-conjugate $\mathscr{D}$-cuts by Lemma 1. $\square$

The next task is to find a similar lower bound in the case when $M \models \mathrm{Th}(\mathbb{N})$. In fact, we will prove something slightly more general. We first need a technical lemma.

Lemma 5. If $P_Y(M)$ is true, then there is a strictly increasing function $H\colon M \to M$ definable in M without parameters such that

$$\exists x \in M\, \forall k \in M\;\; Y(H(x+k), H(x+k+1)) > \mathbb{N}.$$

Proof. In the case when $M \models \forall n \forall x \exists y Y(x,y) \geqslant n$, let H to be the function defined recursively by

$$H(0) = 0 \text{ and } \forall z \in M\; H(z+1) = Y_{z+1}(H(z)).$$

In the case when $M \models \exists n \exists x \forall y Y(x,y) < n$, define H by

$$H(0) = 0 \text{ and } \forall z \in M\; H(z+1) = Y_n(H(z)),$$

where n is the maximum $m \in M$ such that $M \models \forall x \exists y Y(x,y) \geqslant m$. $\square$

Theorem 6. (a) If $P_Y(M)$ is false, then no $\mathscr{D}$-cut contains $\overline{\mathrm{cl}}(\varnothing)$.

(b) If $P_Y(M)$ is true, then there are at least countably infinitely many non-conjugate $\mathscr{D}$-cuts containing $\overline{\mathrm{cl}}(\varnothing)$.

Proof. (a) Suppose $P_Y(M)$ is false, then $\exists n \in \mathbb{N}\; M \models \exists x \forall y Y(x,y) < n$. Take such $n \in \mathbb{N}$ and the least $x \in M$ such that $M \models \forall y Y(x,y) < n$. Then $x \in \mathrm{cl}(\varnothing)$ and there is no Y-interval above x since $n \in \mathbb{N}$. So, there cannot be any $\mathscr{D}$-cut above $\overline{\mathrm{cl}}(\varnothing)$.

(b) Suppose $P_Y(M)$ is true, and let H be the function guaranteed by Lemma 5. Pick $x > \overline{\mathrm{cl}}(\varnothing)$ such that $([H(x+k), H(x+k+1)])_{k \in \mathbb{N}}$ is a sequence of Y-intervals. By the density of $\mathscr{D}$ in Z_Y^M, take a sequence $(I_k)_{k \in \mathbb{N}}$ of $\mathscr{D}$-cuts such that

$$\forall k \in \mathbb{N}\;\; I_k \in [H(x+k), H(x+k+1)].$$

Suppose $i, j \in \mathbb{N}$ such that I_i and I_j are conjugate. Then i and j are respectively the biggest $k \in \mathbb{N}$ such that $H(x+k)$ is in I_i and the biggest $k \in \mathbb{N}$ such that $H(x+k)$ is in I_j. It follows that for sufficiently large $m \in \mathbb{N}$,

$$i \equiv j \mod m$$

by conjugacy of I_i and I_j, and so $i = j$. This shows that the cuts in $(I_k)_{k \in \mathbb{N}}$ are mutually non-conjugate. $\square$

In general, these lower bounds may not be tight. However, they are met in the case of generic cuts. Notice that if M is arithmetically saturated, then there are uncountably many generic cuts [1, Theorem 5.7].

Corollary 7. If M is arithmetically saturated, then there are exactly countably infinitely many conjugacy classes of generic cuts in M.

Proof. Recall that if two generic cuts are in the same 0-small 0-constant Y-interval, then they are conjugate [1, Theorem 5.6]. By the countability of M, this implies that there can at most be countably infinitely many conjugacy classes of generic cuts in M.

On the other hand, note that by the non-degeneracy of Y, it is not possible that $\neg P_Y(M)$ and $M \models \mathrm{Th}(\mathbb{N})$. So, by Theorems 4 and 6, and the fact that generic cuts are dense in Z_Y^M [1, Theorem 5.7], there are exactly countably infinitely many conjugacy classes of generic cuts in M. □

References

[1] Richard Kaye. Generic cuts in models of arithmetic. To appear. Available on the internet at `http://web.mat.bham.ac.uk/R.W.Kaye/papers/generic`, February 15, 2007.