The model theory of generic cuts

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We survey some properties of the recently discovered notion of *generic cuts* [6, 8], and discuss the model-theoretic context of these results for theories of pairs (M, I) where I is a generic cut of a model M of Peano arithmetic.

The main new results are that such pairs possess certain model completeness properties (Section 4), and they are existentially closed in a suitable category (Section 5).

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1 Preliminaries

For background in model theory, see Hodges [4]. Unless otherwise stated, we follow the notation in the books by Kaye [5] and by Kossak–Schmerl [12]. We fix some notation and repeat a few more relevant definitions here.

The usual language for arithmetic $\{0, 1, +, \times, <\}$ is denoted by $\mathscr{L}_\mathrm{A}$. Let $\mathscr{L}_\mathrm{Sk}$ denote the Skolemised language for arithmetic, i.e., $\mathscr{L}_\mathrm{Sk}$ contains, in addition to the symbols in $\mathscr{L}_\mathrm{A}$, a function symbol $f_\theta(\bar{x})$ for each formula $\theta(\bar{x}, y) \in \mathscr{L}_\mathrm{A}$, intending to mean the least y satisfying $\theta(\bar{x}, y)$. By 'definable', we always mean 'definable with parameters'. All our types can only contain finitely many parameters. We write $\mathsf{Q}x \ldots$ for 'there are cofinally many x such that $\ldots$'. Peano arithmetic is abbreviated as PA.

A *cut* of a model of arithmetic is a nonempty proper initial segment I that has no maximum element. We write $I \subseteq_\mathrm{e} M$ for 'I is a cut of M'. If I is, in addition, an elementary substructure of the model, then we say that I is an *elementary cut*.

Let $\bar{c} \in M \models \mathrm{PA}$. We denote by $\mathrm{Aut}(M)$ the automorphism group of M, and by $\mathrm{Aut}(M, \bar{c})$ the pointwise stabiliser of $\bar{c}$ in $\mathrm{Aut}(M)$. If $X \subseteq M$ and f is a function with domain M, then X^f denotes the image of X under f, i.e.,

$$X^f = \{f(x) : x \in X\}.$$

Similarly, we sometimes write x^f for $f(x)$. Two cuts $I, J \subseteq_{\mathrm{e}} M$ are *conjugate* over $\bar{c}$ if there is $g \in \mathrm{Aut}(M, \bar{c})$ such that $I^g = J$.

2 Generic cuts

Generic cuts were discovered by the first author during an axiomatic study on *indicators*. The motivation was to understand structures of the form (M, I), where I is a cut of a model of arithmetic M, in a model-theoretic context. We view a generic cut as representing the majority of cuts in a model. This will be made more precise at the end of this section. For now, let us first review some basic definitions related to generic cuts.

Generic cuts are constructed using a notion of forcing, i.e. a Banach–Mazur game in an appropriate topological space. The open sets are generated by certain *intervals* in a model of arithmetic. If $I \subseteq_{\mathrm{e}} M \models \mathrm{PA}$ and $[a, b] \subseteq M$, then we write $I \in [a, b]$ to mean $a \in I < b$. We will regard an interval both as a set of numbers and as a set of cuts, which of these we are thinking of will be clear from context. The consistency condition for our forcing comes from a neighbourhood system.

Definition. A nonempty set $\mathscr{N}$ of intervals of the form $[x, y]$ from M is a *neighbourhood system* if the following properties hold.

(1) (Nontriviality.) For all $x \in M$ there is some $[x, y] \in \mathscr{N}$.[1]

(2) For each $B \in M$, there exists a recursive Σ_1-type $p(x, y)$ over M such that

$$\forall x, y{<}B \ \Big([x, y] \in \mathscr{N} \Leftrightarrow M \models \bigwedge\!\!\!\bigwedge p(x, y)\Big).$$

(3) $y > (x+1)^2$ for all $[x, y] \in \mathscr{N}$.

(4) If $[x, y] \in \mathscr{N}$ and $z \in M$, then either $[x, z] \in \mathscr{N}$ or $[z, y] \in \mathscr{N}$.

(5) If $[x, y] \in \mathscr{N}$ and $[x', y'] \supseteq [x, y]$, then $[x', y'] \in \mathscr{N}$.

We will be particularly interested in neighbourhood systems that are invariant under the action of the automorphism group of the model. An important example of a neighbourhood system is the set $\mathscr{N}_{\mathrm{elt}}$ of intervals $[a, b]$ for which there is an elementary cut $I \in [a, b]$. This is automorphism-invariant and type-definable in a model of PA; if the model is recursively saturated it is in fact locally type-definable by a recursive Σ_1-type in the sense of (2) above.

Neighbourhood systems arise naturally from indicators, as introduced by Paris and Kirby [9, 10].

Definition. Let $M \models \mathrm{PA}$ be nonstandard. An *indicator* on M is a function $Y\colon M^2 \to M$ that satisfies the following properties.

(1) (Nontriviality.) For all $x \in M$, there exists $y \in M$ such that $Y(x, y) > \mathbb{N}$.

[1] The nontriviality clause has been added to the definition given in our previous paper [8] though it was not required for all the key results there. It implies that the corresponding set of cuts does not have any isolated points.

(2) Y is *piecewise definable*, i.e.,

$$\{\langle x, y, Y(x,y)\rangle : x, y < B\}$$

is definable in M for every $B \in M$.

(3) $y > (x+1)^2$ for all $x, y \in M$ with $Y(x,y) > \mathbb{N}$.

(4) $M \models \forall x, y\ (x \geqslant y \to Y(x,y) = 0)$.

(5) $M \models \forall x, y, x', y'\ (x \leqslant x' \wedge y' \leqslant y \to Y(x', y') \leqslant Y(x,y))$.

(6) If $x, y \in M$ such that $Y(x,y) > \mathbb{N}$ and $z \in [x,y]$, then either $Y(x,z) > \mathbb{N}$ or $Y(z,x) > \mathbb{N}$.

(7) $M \models \forall x, y, z\ \big(Y(x,y) \geqslant z \to \exists x', y'\ (Y(x', y) = z \wedge Y(x, y') = z)\big)$.

We will often treat an indicator as if it were definable. Formally, one has to replace the function with the code of some large enough initial part of it.

An indicator Y clearly gives a neighbourhood system

$$\mathscr{N}_Y = \{[x,y] : Y(x,y) > \mathbb{N}\}.$$

Conversely, provided the model has countable set-theoretic cofinality, we can obtain an indicator Y from any neighbourhood system which is locally defined by recursive types of bounded complexity $n \in \mathbb{N}$. The idea is that $Y(a,b)$ counts the number of formulas in $p(x,y)$ satisfied by the pair a, b, where $p(x,y)$ is as given in clause (2) of the definition of neighbourhood systems. This can be expressed within $\mathscr{L}_\mathrm{A}$ using a Σ_n-complete formula Sat_{Σ_n}. The countable cofinality assumption comes in when one tries to combine a family of 'partial indicators' into a single function — it is not hard to piece two 'partial indicators' together, but we do not see how to deal with infinitely many of them at one time.

Strictly speaking, the material in this paper concerns neighbourhood systems, in the sense that we are only interested in indicators up to the neighbourhood system they determine. However a single indicator function is often considerably easier to work with than a neighbourhood system with its type-definition.

Definition. Let $M \models \mathrm{PA}$ be nonstandard, and Y be an indicator on M.

- A *Y-interval* is an interval $[a,b] \subseteq M$ such that $Y(a,b) > \mathbb{N}$. Double square brackets $[\![\cdot,\cdot]\!]$ will be reserved for denoting Y-intervals.
- A *Y-cut* is a cut $I \subseteq_\mathrm{e} M$ that satisfies $Y(x,y) > \mathbb{N}$ for all intervals $[x,y]$ containing I.
- Y is *invariant* if its neighbourhood system $\mathscr{N}_Y$ is $\mathrm{Aut}(M)$-invariant.

Suppose we have an indicator Y on a nonstandard model $M \models \mathrm{PA}$. We think of Y-intervals as being 'large' enough to 'catch' at least one Y-cut. It is straightforward to verify that the collection of Y-intervals generates a topology on the class of all Y-cuts, and if M is countable, this topological space is homeomorphic to the Cantor set 2^ω (using the nontriviality clause in the definition). Instead of defining generic cuts in terms of forcing, we define them using this topology and the automorphisms of M.

Definition. Let $M \models \mathrm{PA}$ be nonstandard and Y be an indicator on M. A cut $I \subseteq_{\mathrm{e}} M$ is *Y-generic* if it is contained in a class of Y-cuts $\mathscr{G}$ that has the following properties.

(a) $\mathscr{G}$ is *closed under* $\mathrm{Aut}(M)$, i.e., $I^g \in \mathscr{G}$ for all $I \in \mathscr{G}$ and all $g \in \mathrm{Aut}(M)$.

(b) $\mathscr{G}$ is *dense* in the space of Y-cuts, i.e., every Y-interval contains a cut in $\mathscr{G}$.

(c) For each $I \in \mathscr{G}$ and each $\bar{c} \in M$, there is an interval $[a, b] \subseteq M$ containing I in which all cuts in $\mathscr{G}$ are conjugate over $\bar{c}$.

Theorem 2.1. Let M be a countable arithmetically saturated model of PA, and Y be an invariant indicator on M.

(1) There is a unique class $\mathscr{G}$ of Y-cuts that satisfies (a)–(c) in the definition of Y-generic cuts.

(2) The class of Y-generic cuts is the smallest comeagre set in the space of Y-cuts.

The proof of this [8] goes via the combinatorial notion of *pregeneric intervals*, which we do not want to delve into here. (Pregenericity will, nevertheless, make a brief appearance in Theorem 3.7.) Actually, generic cuts were defined in terms of these intervals in our previous paper [8].

Notice part (1) of this theorem implies that around any generic cut, there is an interval in which all generic cuts are conjugate. This is the key property of generic cuts that we will use over and over again.

The name 'generic cuts' comes from part (2) of this theorem. It says that generic cuts satisfy all the properties that are possessed by almost all cuts, when 'almost all' means 'comeagre'.

The previous paper, the definition of generic cuts, and Theorem 2.1 concern neighbourhood systems, indicators and generic cuts in a single model of PA. Some neighbourhood systems, such as $\mathscr{N}_{\mathrm{elt}}$, and their corresponding indicators, make sense across a range of models of PA. When they do we will be interested in their model-theoretic properties, and this is the main focus of this paper. We will get to this in the final section.

3 Saturation

We work with a fixed countable arithmetically saturated model $M \models \mathrm{PA}$ and an invariant indicator Y on M throughout this section.

Before we show how saturated generic cuts are, we need to describe the language involved more explicitly.

Definition. Define $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ to be the language obtained from $\mathscr{L}_{\mathrm{Sk}}$ by adding one new unary predicate symbol $\mathbb{I}$, which is intended to be interpreted as a cut. So with some abuse of notation, we sometimes write $t \notin \mathbb{I}$ as $t > \mathbb{I}$. $\mathrm{PA}^{\mathrm{cut}}$ is the $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$-theory that consists of the axioms of PA in $\mathscr{L}_{\mathrm{Sk}}$, the definitions for the Skolem functions, and an axiom saying $\mathbb{I}$ is a cut.

We divide the $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$-formulas into levels as in usual model theory.

Definition. The formula classes $\forall_n$ and $\exists_n$ in the language $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ will be referred to as Π_n^{cut} and Σ_n^{cut} respectively, for all $n \in \mathbb{N}$.

For example, since $\mathscr{L}_{\mathrm{Sk}}$ is Skolemised, all formulas in $\mathscr{L}_{\mathrm{Sk}}$ are Π_1^{cut}. The theory $\mathrm{PA}^{\mathrm{cut}}$ contains both Π_1^{cut}- and Σ_1^{cut}-formulas. So it is at most Π_2^{cut}. This implies that models of $\mathrm{PA}^{\mathrm{cut}}$ are closed under unions of chains. Using the preservation theorems, it is a straightforward exercise to show that $\mathrm{PA}^{\mathrm{cut}}$ is neither Π_1^{cut} nor Σ_1^{cut}.

Most of the other concepts we will see are either Σ_1^{cut} or Π_1^{cut}. For instance, if $\chi(x, y, \bar{z})$ is an $\mathscr{L}_{\mathrm{A}}$-formula, then 'there is an interval $[x, y]$ containing $\mathbb{I}$ that satisfies $\chi(x, y, \bar{z})$' is Σ_1^{cut}. This is more than just a random example.

Lemma 3.1. Every Σ_1^{cut}-formula is equivalent over $\mathrm{PA}^{\mathrm{cut}}$ to a formula of the form

$$\exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x, y, \bar{z}),$$

where $\chi \in \mathscr{L}_{\mathrm{A}}$.

Proof. Note that every Σ_1^{cut}-formula is equivalent to a finite disjunction of formulas of the form

$$\exists \bar{v}\ \Big(\bigwedge_{i<m} t_i(\bar{v}, \bar{z}) \in \mathbb{I} \wedge \bigwedge_{j<n} s_j(\bar{v}, \bar{z}) > \mathbb{I} \wedge \eta(\bar{v}, \bar{z}) \Big),$$

where the t_i's and s_j's are Skolem functions and η is an $\mathscr{L}_{\mathrm{A}}$-formula, and this displayed formula is equivalent over $\mathrm{PA}^{\mathrm{cut}}$ to

$$\exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \exists \bar{v}\ \big(x = \max_{i<m}\big(t_i(\bar{v}, \bar{z})\big) \wedge y = \min_{j<n}\big(s_j(\bar{v}, \bar{z})\big) \wedge \eta(\bar{v}, \bar{z}) \big). \qquad \square$$

Let us come back to the saturation properties of generic cuts. We start with a notion defined in Kirby's thesis [9].

Definition. The *Y-index* of a cut $I \subseteq_{\mathrm{e}} M$ is defined to be

$$\{ n \in M : (M, I) \models \forall x{\in}\mathbb{I}\ \forall y{>}\mathbb{I}\ Y(x, y) > n \}.$$

The index of a cut is clearly an initial segment of the model.

Proposition 3.2. If I is a Y-generic cut in M, then the Y-index of I is $\mathbb{N}$.

Proof. Every natural number is in the Y-index of I because I is a Y-cut. Take any nonstandard $\nu \in M$. Recall that Y is piecewise definable. Pick any $B \in M \setminus I$, and let $\hat{Y} \in M$ be the code of $\{\langle x, y, Y(x, y)\rangle : x, y < B\}$. Using the genericity of I, let $[a, b]$ be an interval containing I in which all Y-generic cuts are conjugate over $\langle \hat{Y}, \nu \rangle$. Choosing $b = B - 1$ if necessary, we may assume $b < B$. Let $[u, v] \subseteq [a, b]$ such that $\mathbb{N} < Y(u, v) < \nu$, and J be a Y-generic cut in $[u, v]$. By the choice of $[a, b]$, we know that I and J are conjugate over $\langle \hat{Y}, \nu \rangle$. Since

$$(M, J) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \hat{Y}(x, y) \leqslant \nu,$$

the same formula is true in (M, I) too. So ν is not in the Y-index of I. $\square$

Remark. There is a notion that is similar to the index, called the *cofinality*, of a cut. It turns out that the cofinality of a generic cut depends on the indicator chosen. See Theorem 4.13 in Kaye [6] for the details.

This proposition implies a non-saturation property of generic cuts. Recall the following definition: if $\mathscr{L}$ is a recursive language and Γ is a class of $\mathscr{L}$-formulas, then an $\mathscr{L}$-structure $\mathfrak{M}$ is Γ-*recursively saturated* if and only if all recursive types that just consist of formulas in Γ are realised in $\mathfrak{M}$.

Corollary 3.3. If I is a Y-generic cut in M, then (M, I) is not Π_1^{cut}-recursively saturated.

Proof. Consider the type

$$p(v) = \{v > n : n \in \mathbb{N}\} \cup \{\forall x{\in}\mathbb{I}\ \forall y{>}\mathbb{I}\ \hat{Y}(x, y) > v\},$$

where $\hat{Y}$ is a code for some suitable initial part of Y. □

This has consequences on the non-definability of generic cuts, in the sense that no first-order $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$-theory captures only generic cuts.

Corollary 3.4. Let T be a consistent $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$-theory extending $\mathrm{PA}^{\mathrm{cut}}$. Then there is a model $(K, J) \models T$ such that J is not Y-generic for all indicators Y on K.

Proof. Take $(K, J) \models T$ to be countable and recursively saturated. □

In particular no first-order $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$-theory T captures the notion of 'elementary generic', i.e. generic for $\mathscr{N}_{\mathrm{elt}}$.

We will show that the failure of Π_1^{cut}-recursive saturation is best possible for generic cuts. The proof uses a lemma that is worth stating on its own.

Lemma 3.5. Let $[\![a, b]\!]$ be a Y-interval and $p(v)$ be a recursive set of Σ_1^{cut}-formulas that involves only finitely many parameters from M. Then the following are equivalent.

(1) There exists a Y-interval $[\![r, s]\!] \subseteq [\![a, b]\!]$ such that for all Y-cuts I in $[\![r, s]\!]$ the set $p(v)$ is realised in (M, I).

(2) There is a Y-cut $I \in [\![a, b]\!]$ such that $p(v)$ is realised in (M, I).

(3) There is a Y-cut $I \in [\![a, b]\!]$ such that $p(v)$ is finitely satisfied in (M, I).

(4) For each finite $p'(v) \subseteq p(v)$, there is a Y-cut $I \in [\![a, b]\!]$ such that $(M, I) \models \exists v \bigwedge\!\!\!\bigwedge p'(v)$.

Proof. It is clear that $(i) \Rightarrow (i + 1)$ for $i = 1$, 2, and 3. So it suffices to show $(4) \Rightarrow (1)$.

By Lemma 3.1, we may assume

$$p(v) = \{\exists \bar{x}{\in}\mathbb{I}\ \exists \bar{y}{>}\mathbb{I}\ \theta_i(v, \bar{x}, \bar{y}, \bar{c}) : i \in \mathbb{N}\},$$

where $\bar{c} \in M$ and $\theta_0, \theta_1, \ldots \in \mathscr{L}_{\mathrm{A}}$. Consider the recursive set

$$\begin{aligned} q(v, r, s) = {} & \{Y(r, s) > n : n \in \mathbb{N}\} \cup \{a \leqslant r \wedge s \leqslant b\} \\ & \cup \{\exists \bar{x}{<}r\ \exists \bar{y}{>}s\ \theta_i(v, \bar{x}, \bar{y}, \bar{c}) : i \in \mathbb{N}\} \end{aligned}$$

of $\mathscr{L}_{\mathrm{A}}$-formulas. If this is realised in M, then (1) is true. So by recursive saturation, it suffices to show $q(v, r, s)$ is finitely satisfied in M. We can prove

$$M \models \exists v\ \exists [r, s]{\subseteq}[\![a, b]\!]\ \Big(Y(r, s) > n \wedge \bigwedge\!\!\!\bigwedge_{i<n} \exists \bar{x}{<}r\ \exists \bar{y}{>}s\ \theta_i(v, \bar{x}, \bar{y}, \bar{c})\Big)$$

for every $n \in \mathbb{N}$ from the hypothesis that (4) holds. □

Proposition 3.6. If I is a Y-generic cut in M, then (M, I) is Σ_1^{cut}-recursively saturated.

Proof. Let $p(v)$ be a Σ_1^{cut}-recursive type that is finitely satisfied in (M, I), and $\bar{c} \in M$ be the parameters that appear in $p(v)$. Using genericity, pick an interval $[a, b] \subseteq M$ containing I in which all Y-generic cuts are conjugate over $\bar{c}$. By the previous lemma, we can find a Y-interval $[\![r, s]\!] \subseteq [\![a, b]\!]$ in which all Y-cuts J make $p(v)$ realised in (M, J). Inside such an interval, there is a Y-generic cut that is conjugate to I over $\bar{c}$. So $p(v)$ is realised in (M, I) too. □

Strangely, generic cuts seem to possess more than what Σ_1^{cut}-recursive saturation can offer. For example, the following apparently needs Π_1^{cut}-recursive saturation. The extra bit comes from the strength of $\mathbb{N}$.

Theorem 3.7. Let I be a Y-generic cut in M. Then for every $\bar{c} \in M$, there exists an interval $[a, b]$ containing I such that for every $\mathscr{L}_{\mathrm{A}}$-formula $\chi(x, y, \bar{z})$,

$$(M, I) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x, y, \bar{c}) \rightarrow \exists x{<}a\ \exists y{>}b\ \chi(x, y, \bar{c}).$$

Proof. Let $[u, v]$ be an interval around I in which all Y-generic cuts are conjugate over $\bar{c}$. Using recursive saturation, find $f \in M$ coding a function $\mathbb{N} \rightarrow M$ such that

$$f(\chi) = (\max n)\big(\exists [r, s]{\subseteq}[u, v]\ (Y(r, s) \geqslant n \wedge \exists x{<}r\ \exists y{>}s\ \chi(x, y, \bar{c}))\big),$$

for all $\mathscr{L}_{\mathrm{A}}$-formulas $\chi \in \mathbb{N}$. Let $d \in M \setminus \mathbb{N}$ such that

$$f(\chi) > \mathbb{N} \Leftrightarrow f(\chi) > d,$$

for all $\chi \in \mathbb{N}$. Such d exists because $\mathbb{N}$ is strong in M. We claim that for every $\chi \in \mathbb{N}$, we have $f(\chi) > d$ if and only if

$$(M, I) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x, y, \bar{c}).$$

This suffices because we can then write our type as

$$p(a, b) = \{a \in \mathbb{I} \wedge \mathbb{I} < b\} \cup \{f(\chi) > d \rightarrow \exists x{<}a\ \exists y{>}b\ \chi(x, y, \bar{c}) : \chi \in \mathscr{L}_{\mathrm{A}}\},$$

which is recursive and Σ_1^{cut}.

Let $\chi \in \mathbb{N}$. If $f(\chi) \leqslant d$, then $f(\chi) \in \mathbb{N}$, and so

$$(M, I) \models \forall x{\in}\mathbb{I}\ \forall y{>}\mathbb{I}\ \neg\chi(x, y, \bar{c})$$

since I is a Y-cut. Conversely, suppose $f(\chi) > d$. Let $[r, s] \subseteq [u, v]$ such that $Y(r, s) = f(\chi) > \mathbb{N}$ and $M \models \exists x{<}r\ \exists y{>}s\ \chi(x, y, \bar{c})$. By our choice of $[u, v]$, the cut I is conjugate over $\bar{c}$ to some Y-generic cut $J \in [r, s]$. For any such J, we have

$$(M, J) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x, y, \bar{c}).$$

Therefore, this formula is also true in (M, I). □

We present two applications of this theorem, the first of which is easy.

Corollary 3.8. Let I be a Y-generic cut in M. Then for all $\bar{c} \in M$, there exists $a \in I$ such that for every $\mathscr{L}_{\mathrm{A}}$-formula $\eta(v, \bar{z})$,

$$(M, I) \models \exists x{\in}\mathbb{I}\ \forall v{\in}\mathbb{I}\ \big(v > x \rightarrow \eta(v, \bar{c})\big) \rightarrow \exists x{<}a\ \forall v{\in}\mathbb{I}\ \big(v > x \rightarrow \eta(v, \bar{c})\big).$$

Proof. Let $\chi(x, y, \bar{c})$ be $\forall v{\in}[x, y]\ \eta(v, \bar{c})$. □

The second one seems slightly more tricky. Recall first the following.

Definition. Let $\mathfrak{M}$ be a structure in some first-order language $\mathscr{L}$, and $\bar{c}, r \in \mathfrak{M}$. The *existential type* of r over $\bar{c}$, denoted by $\mathrm{etp}_{\mathfrak{M}}(r/\bar{c})$, is defined to be

$$\{\phi(w, \bar{c}) \in \exists_1 : \mathfrak{M} \models \phi(r, \bar{c})\}.$$

As is commonly done in models of PA, we regard each element of M as an M-finite set here. See our previous paper [7] for details. Our consequence of Theorem 3.7 now follows.

Corollary 3.9. Let I be a Y-generic cut in M. Then the existential type of any tuple $\bar{c}$ in (M, I) is *coded*, i.e., there exists $t \in M$ such that for all $\phi(\bar{w}) \in \Sigma_1^{\mathrm{cut}}$,

$$(M, I) \models \ulcorner\phi\urcorner \in t \leftrightarrow \phi(\bar{c}).$$

Proof. By Lemma 3.1, it suffices to consider only Σ_1^{cut}-formulas of the form

$$\exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x, y, \bar{z}),$$

where $\chi \in \mathscr{L}_{\mathrm{A}}$. By Theorem 3.7, we can find an interval $[a, b]$ containing I such that

$$(M, I) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x, y, \bar{c}) \leftrightarrow \exists x{<}a\ \exists y{>}b\ \chi(x, y, \bar{c}),$$

for every $\mathscr{L}_{\mathrm{A}}$-formula $\chi(x, y, \bar{z})$. So recursive saturation of M implies that

$$p(t) = \{\ulcorner\exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x, y, \bar{z})\urcorner \in t \leftrightarrow \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x, y, \bar{c}) : \chi \in \mathscr{L}_{\mathrm{A}}\}$$

is realised in M. □

4 Model completeness

Throughout this section, we work with a fixed countable arithmetically saturated model $M \models \mathrm{PA}$, an invariant indicator Y on M, and a Y-generic cut I in M.

In our previous paper [8], we established a weak quantifier elimination result. We reformulate this theorem here in more model-theoretic terms, and list some consequences related to model completeness.

Theorem 4.1. Let $\bar{c} \in M$ and J be a Y-generic cut in M. If $\mathrm{etp}_{(M,I)}(\bar{c}) = \mathrm{etp}_{(M,J)}(\bar{c})$, then $(M, I, \bar{c}) \cong (M, J, \bar{c})$.

Proof sketch. Without loss, assume $I < J$. Using genericity, pick intervals $[a, b]$ and $[u, v]$ containing I and J respectively in which all Y-generic cuts are conjugate over $\bar{c}$. Let $x \in I$ that is above a. We claim that there is $h \in \mathrm{Aut}(M, \bar{c})$ such that $x^h \in [u, v]$, which will give us what we want. By recursive saturation, it suffices to show that the $\mathscr{L}_{\mathrm{A}}$-type of x over $\bar{c}$ is finitely satisfied in $[u, v]$. Let $\theta(x, \bar{w}) \in \mathscr{L}_{\mathrm{A}}$ such that $M \models \theta(x, \bar{c})$. Then $(M, I) \models \mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{c})$ by (the proof of) Corollary 3.8. Notice $\mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{w})$ is Π_1^{cut}, and $\neg\mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{w}) \notin \mathrm{etp}_{(M,I)}(\bar{c})$. Thus $\neg\mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{w}) \notin \mathrm{etp}_{(M,J)}(\bar{c})$, which means $(M, J) \models \mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{c})$. In particular, there is $y \in J$ above u such that $M \models \theta(y, \bar{c})$. □

Corollary 4.2. For any $\bar{c}, r, s \in M$, if $\mathrm{etp}_{(M,I)}(r/\bar{c}) = \mathrm{etp}_{(M,I)}(s/\bar{c})$, then $(M, I, \bar{c}, r) \cong (M, I, \bar{c}, s)$.

Proof. Suppose $\mathrm{etp}_{(M,I)}(r/\bar{c}) = \mathrm{etp}_{(M,I)}(s/\bar{c})$. Using recursive saturation, let $g \in \mathrm{Aut}(M, \bar{c})$ such that $s^g = r$. Setting $J = I^g$ gives $(M, I, \bar{c}, s) \cong (M, J, \bar{c}, r)$, and so by hypothesis,

$$\mathrm{etp}_{(M,I)}(\bar{c}, r) = \mathrm{etp}_{(M,I)}(\bar{c}, s) = \mathrm{etp}_{(M,J)}(\bar{c}, r).$$

The previous theorem then does the rest. □

Definable elements in models of $\mathrm{PA}^{\mathrm{cut}}$ were studied in Kossak–Bamber [11]. Corollary 4.2 gives us more information about these definable elements in the case of generic cuts.

Corollary 4.3. All $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$-definable elements of (M, I) are Σ_1^{cut} definable with the same parameters.

Proof. Let $a, \bar{c} \in M$ and $\theta(w, \bar{z}) \in \mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ such that a is the unique element w that satisfies $\theta(w, \bar{c})$ in (M, I). Consider $p(w) = \mathrm{etp}_{(M,I)}(a/\bar{c})$. If $a' \in M$ that satisfies all of $p(w)$ in (M, I), then $(M, I, \bar{c}, a) \cong (M, I, \bar{c}, a')$ by Corollary 4.2, and so $a = a'$ because they both satisfy $\theta(w, \bar{c})$. It follows that

$$q(w) = p(w) \cup \{w \neq a\}$$

is not realised in (M, I). The set $q(w)$ can be rewritten as a recursive set of Σ_1^{cut}-formulas using Corollary 3.9. So by Σ_1^{cut}-recursive saturation, this set is not finitely satisfied in (M, I). Therefore, there is $\phi(w, \bar{c}) \in p(w)$ such that

$$(M, I) \models \forall w \ (\phi(w, \bar{c}) \to w = a),$$

as required. □

Remark. Typical Σ_1^{cut}-definable elements are

$$(\max w \in \mathbb{I})(\theta(w, \bar{c})) \quad \text{and} \quad (\min w > \mathbb{I})(\theta(w, \bar{c})),$$

where $\theta(w, \bar{c})$ is in $\mathscr{L}_{\mathrm{A}}$.

Question 4.4. Can one make the map $\theta(w, \bar{z}) \mapsto \phi(w, \bar{z})$, which is given by the proof above, independent of $\bar{c}$?

Recall that a definition of model completeness says: a theory T is *model complete* if and only if every $\forall_1$-formula is equivalent modulo T to a $\exists_1$-formula.

Corollary 4.5. For every Π_1^{cut}-formula $\theta(w, \bar{z})$ and every $\bar{c} \in M$, there exists a set $\Phi(w, \bar{z})$ of Σ_1^{cut}-formulas such that

$$(M, I) \models \forall w \ \Big(\theta(w, \bar{c}) \leftrightarrow \bigvee\!\!\!\bigvee \Phi(w, \bar{c})\Big).$$

Proof. Let $A = \{w \in M : M \models \theta(w, \bar{c})\}$. For the moment, work with a fixed $a \in A$. Set $p_a(w) = \mathrm{etp}_{(M,I)}(a/\bar{c})$. By Corollary 4.2,

$$(M, I) \models \forall w \ \Big(\bigwedge\!\!\!\bigwedge p_a(w) \to \theta(w, \bar{c})\Big).$$

Therefore, the set $p_a(w)\cup\{\neg\theta(w,\bar{c})\}$ is not realised in (M,I). As in the previous proof, we use Σ_1^{cut}-recursive saturation to pick $\phi_a(w,\bar{c})\in p_a(w)$ such that

$$(M,I)\models\forall w\ (\phi_a(w,\bar{c})\to\theta(w,\bar{c})).$$

For every $a\in A$, we can find such a formula $\phi_a(w,\bar{c})$. It can be verified that $\Phi(w,\bar{z})=\{\phi_a(w,\bar{z}):a\in A\}$ does what we want. □

Question 4.6. Is the corollary above true for all $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$-formulas $\theta(w,\bar{z})$?

In Corollary 4.5 above, we cannot guarantee that the set $\Phi(w,\bar{z})$ is finite. This can be proved using Corollary 3.3 and Proposition 3.6. Therefore, $\mathrm{Th}(M,I)$ is not model complete.

5 Existential closure

In this section, we show that generic cuts are existentially closed in a suitable category. Existentially closed models of arithmetic were studied by Goldrei, Macintyre, Simmons [2, 13], Hirschfeld, Wheeler [3], and others in the 1970s. A more recent reference is Adamowicz–Bigorajska [1].

Recall that an *existentially closed* model of a theory T is a model $\mathfrak{M}\models T$ such that whenever $\mathfrak{K}\models T$ that extends $\mathfrak{M}$, if $\sigma(\bar{z})\in\exists_1$ and $\bar{c}\in\mathfrak{M}$, then $\mathfrak{K}\models\sigma(\bar{c})$ implies $\mathfrak{M}\models\sigma(\bar{c})$. This property requires us to consider indicators across several models.

Definition. Let $\mathbf{K}$ be a class of models of PA. An *(invariant) indicator* over $\mathbf{K}$ is a sequence $(Y(x,y)=n)_{n\in\mathbb{N}}$ of $\mathscr{L}_{\mathrm{A}}$-formulas with a recursive definition that can be extended to $(Y(x,y)=n)_{n\in M}$ to obtain an (invariant) indicator on M for every $M\in\mathbf{K}$. For simplicity, we will often write an indicator over $\mathbf{K}$ as Y instead of $(Y(x,y)=n)_{n\in\mathbb{N}}$. For an indicator Y over some class of models of PA, define $\mathrm{PA}_Y^{\mathrm{cut}}=\mathrm{PA}^{\mathrm{cut}}\cup\{\forall x{\in}\mathbb{I}\ \forall y{>}\mathbb{I}\ Y(x,y)\geqslant n:n\in\mathbb{N}\}$, where $Y(x,y)\geqslant n$ means $\bigwedge\!\!\!\bigwedge_{i<n}Y(x,y)\neq i$.

Note that $\mathrm{PA}_Y^{\mathrm{cut}}$ is Π_2^{cut}.

Almost all indicators that appeared in the literature are definable, and their definitions make sense over all models of PA. This allows one to prove independence results for PA using indicators. See Chapter 14 in Kaye [5] for an exposition on this line.

There is a uniform way of defining an indicator Y_{elt} for the neighbourhood system $\mathscr{N}_{\mathrm{elt}}$ in the sense above over the class of all recursively saturated models of PA. For this Y the theory $\mathrm{PA}_Y^{\mathrm{cut}}$ is the theory of models of PA with a distinguished proper elementary cut.

We need one technical lemma before the theorem.

Lemma 5.1. Fix a countable arithmetically saturated model $M\models\mathrm{PA}$, an invariant indicator Y on M, and a Y-generic cut I in M. If Θ is an $\mathscr{L}_{\mathrm{A}}$-definable function under which I is closed, then there is $n\in\mathbb{N}$ such that for all sufficiently large $x\in I$, we have $Y(x,\Theta(x))<n$.

Proof. Find an interval $[a,b]$ around I in which all Y-generic cuts are conjugate over the parameters $\bar{c}$ needed to define Θ. Suppose I is closed under Θ, but for all $n\in\mathbb{N}$, there are cofinally many $x\in I$ such that $Y(x,\Theta(x))\geqslant n$. Then

$$M\models\exists x{\in}[a,b]\ (\Theta(x)\in[a,b]\wedge Y(x,\Theta(x))\geqslant n),$$

for all $n \in \mathbb{N}$. Using recursive saturation, find nonstandard $\nu \in M$ and $x \in [a, b]$ such that

$$M \models \Theta(x) \in [a, b] \wedge Y(x, \Theta(x)) \geqslant \nu.$$

Note $[x, \Theta(x)]$ is a Y-interval. So it contains a Y-generic cut, say J. However J is not closed under Θ, and hence I cannot be conjugate to J over $\bar{c}$. This contradicts our choice of $[a, b]$. □

Theorem 5.2. If M is a countable arithmetically saturated model of PA, and I is a Y-generic cut in M, then (M, I) is an existentially closed model of $\mathrm{PA}_Y^{\mathrm{cut}}$.

Proof. Let (K, J) be an extension of (M, I) that satisfies $\mathrm{PA}_Y^{\mathrm{cut}}$. Notice we must have $M \prec K$ since $\mathscr{L}_{\mathrm{Sk}}$ is Skolemised. Suppose $(K, J) \models \sigma(c)$, where $c \in M$ and $\sigma(z) \in \Sigma_1^{\mathrm{cut}}$. By Lemma 3.1, we may assume $c > I$ and $\sigma(z)$ is of the form

$$\exists x\ \exists y\ (\theta(x, y, z) \wedge x \in \mathbb{I} \wedge y \notin \mathbb{I}),$$

where $\theta \in \mathscr{L}_{\mathrm{A}}$. Define

$$\Theta(x) = \begin{cases} (\max y)(\theta(x, y, c)), & \text{if it exists;} \\ c, & \text{if } \forall u\ \exists y{>}u\ \theta(x, y, c); \\ 0, & \text{if } \neg\exists y\ \theta(x, y, c). \end{cases}$$

The fact that $(K, J) \models \sigma(c)$ implies that J is not closed under Θ.

Suppose first that we have $a \in I$ such that $\Theta(x) > J$ for some $x < a$ in J. Then

$$(K, J) \models (\min b)(\forall x{<}a\ (\Theta(x) \leqslant b)) > \mathbb{I}.$$

This transfers to (M, I), and so $(M, I) \models \sigma(c)$.

Suppose next that such an a cannot be found. Then I is closed under Θ. By the previous lemma, there is $n \in \mathbb{N}$ such that for all sufficiently large $x \in I$,

$$(M, I) \models Y(x, \Theta(x)) < n.$$

By overspill there is $[a, b] \subseteq M$ containing I such that

$$M \models \forall x \in [a, b]\ Y(x, \Theta(x)) < n.$$

This transfers to (K, J) as $M \prec K$. However, such a statement cannot be true in (K, J) because J is a Y-cut under which Θ is not closed. □

It follows immediately that if I is a generic cut in a model $M \models \mathrm{PA}$, then the existential type of any $c \in (M, I)$ is *maximal* over $\mathrm{PA}_Y^{\mathrm{cut}}$, i.e., for all Σ_1^{cut}-formulas $\sigma(z)$, either $\mathrm{PA}_Y^{\mathrm{cut}} + \mathrm{etp}_{(M,I)}(c) \vdash \sigma(c)$ or $\mathrm{PA}_Y^{\mathrm{cut}} + \mathrm{etp}_{(M,I)}(c) \vdash \neg\sigma(c)$.

Question 5.3. If I is a cut of a countable arithmetically saturated model $M \models \mathrm{PA}$ such that for all $\bar{c} \in M$ the existential type $\mathrm{etp}_{(M,I)}(\bar{c})$ is maximal over $\mathrm{PA}_Y^{\mathrm{cut}}$, then is the cut I always Y-generic for some indicator Y on M?

A strengthening of existential closure is *existential universality*, which has been around in the literature for some time [3]. Recall that an *existentially universal* model of a theory T is a model $\mathfrak{M} \models T$ such that whenever $\mathfrak{K} \models T$ that extends $\mathfrak{M}$, if $p(v, \bar{z})$ is a set of $\exists_1$-formulas and $\bar{c} \in \mathfrak{M}$, then $\mathfrak{K} \models \exists v \bigwedge\!\!\!\bigwedge p(v, \bar{c})$

implies $\mathfrak{M} \models \exists v \bigwedge p(v, \bar{c})$. No countable nonstandard model $M \models \mathrm{PA}$ is existentially universal: just consider

$$p(v) = \{i \in v : i \in S\} \cup \{i \notin v : i \in \mathbb{N} \setminus S\},$$

where $S \subseteq \mathbb{N}$ that is not in $\mathrm{SSy}(M)$. For the same reason, no countable model of $\mathrm{PA}^{\mathrm{cut}}$ is existentially universal. To avoid this, we relax the definition a bit.

Definition. Let T be a theory in a recursive language $\mathscr{L}$. Then a *recursively existentially universal* model of T is a model $\mathfrak{M} \models T$ such that whenever $\mathfrak{K} \models T$ that extends $\mathfrak{M}$, if $p(v, \bar{z})$ is a recursive set of $\exists_1$-formulas and $\bar{c} \in \mathfrak{M}$, then $\mathfrak{K} \models \exists v \bigwedge p(v, \bar{c})$ implies $\mathfrak{M} \models \exists v \bigwedge p(v, \bar{c})$.

Corollary 5.4. Fix an invariant indicator Y over some class of models of PA. If M is a countable arithmetically saturated model of PA, and I is a Y-generic cut in M, then (M, I) is a recursively existentially universal model of $\mathrm{PA}_Y^{\mathrm{cut}}$.

Proof. Take a tuple $\bar{c} \in M$ and a recursive set of Σ_1^{cut}-formulas $p(v, \bar{z})$ that is realised in some extension of (M, I) satisfying $\mathrm{PA}_Y^{\mathrm{cut}}$. Then $p(v, \bar{c})$ is finitely satisfied in (M, I) by existential closure. So Σ_1^{cut}-recursive saturation guarantees that $p(v, \bar{c})$ is realised in (M, I). □

It is standard model-theoretic fact that all embeddings between existentially closed models are $\exists_2$-elementary. This applies to generic cuts too. The previous corollary suggests that generic cuts may actually satisfy something stronger. Recall that all embeddings between existentially universal models are elementary [2, 3]. We only have a partial result towards this.

Corollary 5.5. Let Y be an invariant indicator over a class $\mathbf{K}$ of models of PA, and let I, J be Y-generic cuts in countable arithmetically saturated models $M, K \in \mathbf{K}$ respectively. If $\mathrm{SSy}(M) = \mathrm{SSy}(K)$, then every embedding $(M, I) \to (K, J)$ is elementary.

Proof. Let $h\colon (M, I) \to (K, J)$ be an embedding, $\bar{c} \in M$ and $\bar{d} = \bar{c}^h$. Suppose $(K, J) \models \exists w\ \theta(w, \bar{d})$ where $\theta \in \mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$, and take $s \in K$ with $(K, J) \models \theta(s, \bar{d})$. Then $p(w, \bar{d}) = \mathrm{etp}_{(K,J)}(s/\bar{d})$ is coded in K by Corollary 3.9, and hence it is coded in M too. Also $\mathrm{etp}_{(M,I)}(\bar{c}) = \mathrm{etp}_{(K,J)}(\bar{d})$, since the inclusion $\subseteq$ is obvious and $\mathrm{etp}_{(M,I)}(\bar{c})$ is maximal by Theorem 5.2. Thus $p(w, \bar{c})$ is finitely satisfied in (M, I) and by Proposition 3.6 is realised in (M, I), by some $r \in M$ say. Let $s' = r^h \in K$. Then $\mathrm{etp}_{(K,J)}(s/\bar{d}) = \mathrm{etp}_{(K,J)}(s'/\bar{d})$ by maximality again and so $\mathrm{tp}_{(K,J)}(s/\bar{d}) = \mathrm{tp}_{(K,J)}(s'/\bar{d})$ by Corollary 4.2. Therefore $(K, J) \models \theta(s', \bar{d})$ as required. □

Question 5.6. Can the assumption $\mathrm{SSy}(M) = \mathrm{SSy}(K)$ in Corollary 5.5 be removed?

We also ask a related question that is specific to Y_{elt}-generic cuts.

Question 5.7. Let I, J be elementary generic cuts in countable arithmetically saturated models $M, K \models \mathrm{PA}$ respectively. Is it necessarily the case that if $(M, \mathbb{N}) \equiv (K, \mathbb{N})$, then $(M, I) \equiv (K, J)$?

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