Truth in generic cuts

Richard Kaye and Tin Lok Wong

School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom

Abstract

In an earlier paper (*MLQ* 54, 129–144) the first author initiated the study of generic cuts of a model of Peano arithmetic relative to a notion of an indicator in the model. This paper extends that work. We generalise the idea of indicator to a related *neighbourhood system*; this allows the theory to be extended to one that includes the case of elementary cuts. Most results transfer to this more general context, and in particular we obtain the idea of a generic cut relative to a neighbourhood system, which is studied in more detail. The main new result on generic cuts presented here is a description of truth in the structure (M, I), where I is a generic cut of a model M of Peano arithmetic. The special case of elementary generic cuts provides a partial answer to a question of Kossak (*Notre Dame J. Formal Logic* 36, 519–530).

Key words: Generic cuts, Peano arithmetic
1991 MSC: 03C62, 03H15

1 Introduction

The first author has introduced the idea of a generic cut of a model M of Peano arithmetic [2]. His paper, which we refer to as GCMA for convenience, considers the set of cuts or initial segments of a model of arithmetic as a topological space. An indicator serves to select a subspace of this space and give an idea of distance. A generic cut, relative to the chosen indicator, is an element of this subspace which is a member of each comeagre subset that is invariant under automorphisms of the original model M. It was shown in GCMA that generic cuts exist in all countable arithmetically saturated models of PA, and some of their properties were studied.

Email addresses: R.W.Kaye@bham.ac.uk (Richard Kaye), wongtl@maths.bham.ac.uk (Tin Lok Wong).

Preprint submitted to Annals of Pure and Applied Logic *16 October 2009*

The first aim of this paper is to generalise this to a setting that admits the case of elementary cuts as a special case. In Section 2, we give the basic definitions, namely that of a *neighbourhood system*, and that of a *species*. A neighbourhood system is an abstraction of the topological information obtained from an indicator, together with some conditions on definability in the model. A species is a set of cuts that can be indicated by a neighbourhood system. The main relaxation in the definitions here is in using classes or class functions in the usual sense of these words in models of arithmetic, instead of sets and functions which are definable outright.

In Section 3 we set up the topology in which we will work. The major step is proving that any closed species in a countable model is homeomorphic to 2^ω or $2^\omega + 1$ where 2^ω denotes the Cantor set. This enables us to apply the Baire Category Theorem to and play Banach–Mazur games on our space to obtain information about *enforceable* properties. We go on to define the central notion of this paper, that of a generic cut. Although we are not in a position to prove existence theorems at this stage, we do prove a theorem showing the existence of generic cuts under rather general hypotheses (Theorem 3.10) that will be particularly useful in motivating the results in Section 5 and Section 6.

Section 4 gives examples of enforceable properties and serves to provide a list of properties enjoyed by generic cuts when they do exist. Most of this section is rather similar to results in GCMA and serve to illustrate that this work lifts easily to the more general situation we are now in.

Section 5 gives the existence theorems for generic cuts in countable arithmetically saturated models of arithmetic. Once again, the proof models that in GCMA, but a more elegant approach turns out to be possible by looking at multi-variable versions of homogeneity notions in GCMA. Also, we have taken the time to extend this argument by showing the necessity of arithmetic saturation, and to analyse the proof into its finitistic core, with a view to extracting information about the true statements in the structure (M, I) where I is generic.

Section 6 studies how generic cuts behave under the action of the automorphism group of the model. The back-and-forth system that we took from GCMA is what most our results there are based on. A few new conjugacy and non-conjugacy properties are proved, including a characterisation of when two generic cuts are conjugate. We also give here a weak quantifier elimination result, the main theorem in this paper. It says that if I is a generic cut of a model M of PA, then the orbit of an element of M under the action of $\mathrm{Aut}(M, I)$ is completely determined by classes that are relatively low in the formula hierarchy.

We conclude the paper in Section 7 by gathering together various facts about *elementary generic cuts* and surveying the relationships of them to the elementary cuts that appeared in the literature. In particular, we show that elementary generic cuts give new examples of *free cuts*, a notion introduced by Roman Kossak. This partially answers a question raised by him on the cardinality of orbits of free cuts, and suggests new ways to tackle his other problems too.

Except where specifically noted, the notation used in this paper is standard, and follows that in GCMA, Kaye [1] and Kossak–Schmerl [9]. It is sometimes helpful to consider models of Peano arithmetic as models of finite set theory via the usual Ackermann interpretation [3]. We assume some knowledge of semiregular, regular and strong cuts, the basic properties of which can be found in Kirby–Paris [5] and the book by Kossak and Schmerl already mentioned. Oxtoby [13] contains some useful background on Baire category.

Most of the results in this paper first appeared in the second author's qualifying MPhil dissertation at the University of Birmingham.

2 Neighbourhood systems and species of cuts

Throughout this paper, M *is a nonstandard model of* PA. We write $\mathscr{L}_{\mathrm{A}}$ for the usual first order language $\{+, \times, <, 0, 1\}$ for arithmetic, and $\langle\cdot,\cdot\rangle$ for the usual pairing function in $\mathscr{L}_{\mathrm{A}}$. Let $\mathrm{cl}(\bar{c})$ denote the definable closure of the tuple $\bar{c} \in M$, and $\overline{\mathrm{cl}}(\bar{c})$ the least initial segment of M containing $\mathrm{cl}(\bar{c})$. [1]

We will sometimes adjoin to M a point at infinity, ∞. By definition we have $x < \infty$ and $\infty + x = \infty = \infty - x$ for every $x \in M$. If $B \in M \cup \{\infty\}$, then $M_{<B}$ and $M_{\leqslant B}$ denote respectively the sets

$$\{x \in M : x < B\} \text{ and } \{x \in M : x \leqslant B\}.$$

A *cut* of M is a nonempty initial segment closed under successors. We write $I \subseteq_{\mathrm{e}} M$ to mean 'I is a cut of M'. In distinction to GCMA, we do not require cuts to be $\mathscr{L}_{\mathrm{A}}$ structures here. For $a, b \in M \cup \{\infty\}$ we denote the set

$$\{x \in M : a \leqslant x \leqslant b\}$$

by $[a, b]$. Define

$$\mathcal{C} = \{I : I \subseteq_{\mathrm{e}} M\}$$

[1] There is no universal agreement on this notation at present. Kaye [1] uses $K(M;\bar{c})$ and $I(M;\bar{c})$ for what we call $\mathrm{cl}(\bar{c})$ and $\overline{\mathrm{cl}}(\bar{c})$ respectively, while Kossak–Schmerl [9] uses $\mathrm{Scl}^M(\bar{c})$ for the definable closure of $\bar{c}$ in M.

and

$$\mathcal{S} = \{[a,b] : a,b \in M \cup \{\infty\} \text{ with } a \leqslant b\}.$$

Elements of $\mathcal{S}$ are called *semi-intervals*. A semi-interval is *finite* if neither of its end points is ∞. For $I \in \mathcal{C}$ and $[a,b] \in \mathcal{S}$, we write $I \in [a,b]$ to mean $a \in I < b$.

The automorphism group of M is denoted by $\mathrm{Aut}(M)$. Each automorphism of M extends in the obvious way to $M \cup \{\infty\}$ and to $\mathcal{C}$. All actions by automorphisms are written as superscripts on the right. If $\bar{c} \in M$, then $\mathrm{Aut}(M, \bar{c})$ denotes the pointwise stabiliser of $\bar{c}$ in $\mathrm{Aut}(M)$. Similarly, if $I \in \mathcal{C}$, then $\mathrm{Aut}(M, I)$ denotes the setwise stabiliser of I in $\mathrm{Aut}(M)$.

Definition 2.1. Two cuts I, J are said to be *conjugate over* $\bar{c} \in M$ if $I^g = J$ for some $g \in \mathrm{Aut}(M, \bar{c})$. Two cuts are *conjugate* if they are conjugate over 0. The *conjugacy class* of a cut I is the orbit of I under the action of $\mathrm{Aut}(M)$.

Extending ideas of Paris and Kirby, indicators were defined in GCMA. We set the scene by abstracting the topological information given by an indicator. Recall first the convention [1, Page 146] that, in the model theory of arithmetic, types over a model are not necessarily complete and may contain finitely many parameters from the model. We shall use this terminology throughout.

Definition 2.2. A set $\mathcal{B} \subseteq \mathcal{S}$ is a *neighbourhood system* if

(0) $\mathcal{B}$ is nonempty;
(1) $\mathcal{B}$ is invariant under the action of $\mathrm{Aut}(M)$;
(2) $\forall [a,b] \in \mathcal{B}\ (b > a+1)$;
(3) $\forall [a,b] \in \mathcal{B}\ \forall c \in M\ ([a,c] \in \mathcal{B} \text{ or } [c,b] \in \mathcal{B})$;
(4) $\forall [a,b] \in \mathcal{B}\ \forall [u,v] \in \mathcal{S}\ ([a,b] \subseteq [u,v] \Rightarrow [u,v] \in \mathcal{B})$; and
(5) for every $B \in M$, there exists a recursive Σ_1 type $p(x,y)$ over M such that

$$\forall a,b < B\ \Big([a,b] \in \mathcal{B} \Leftrightarrow M \vDash \bigwedge\!\!\!\bigwedge p(a,b)\Big).$$

If $\mathcal{B}$ is a neighbourhood system and $[a,b] \in \mathcal{B}$, then we say that $[a,b]$ is a $\mathcal{B}$-*interval*, or *interval* if $\mathcal{B}$ is clear from the context. We write $a \ll b$ or $a \ll_{\mathcal{B}} b$ to mean $[a,b]$ is a $\mathcal{B}$-interval. It is also helpful to have a notation for semi-intervals that identifies them as intervals: $[\![a,b]\!]$, or $[\![a,b]\!]_{\mathcal{B}}$ if $\mathcal{B}$ needs to be specified, will always denote a $\mathcal{B}$-interval whereas $[a,b]$ might or might not be an interval.

The following is a basic property of neighbourhood systems.

Proposition 2.3. *Let $\mathcal{B}$ be a neighbourhood system. Then for all finite intervals $[\![a,b]\!]$, there exists $c \in [\![a,b]\!]$ such that $[a,c], [c,b] \in \mathcal{B}$.*

Proof. Let $[\![a,b]\!] \in \mathcal{B}$ be finite, and let $B \in M$ be greater than a, b. Suppose the formulas in the type given by axiom (5) for a neighbourhood system are enumerated recursively in the sequence $(\varphi_n(x,y))_{n\in\mathbb{N}}$ in increasing strength, so that

$$M \vDash \forall x, y < B\ \big(\varphi_{n+1}(x,y) \to \varphi_n(x,y)\big)$$

for all $n \in \mathbb{N}$. By the Σ_1 recursive saturation of M, it suffices to show that for each $n \in \mathbb{N}$, there is $c \in [a,b]$ such that

$$M \vDash \varphi_n(a,c) \wedge \varphi_n(c,b).$$

Pick $n \in \mathbb{N}$ and let $c \in M$ be the least number making $\varphi_n(a,c)$ true. This exists since $[a,b] \in \mathcal{B}$. We can safely assume $c > a+1$ because this holds for large enough n. The minimality of c then implies the falsity of $\varphi_n(a,c-1)$, and so $[a,c-1] \notin \mathcal{B}$. It follows from axioms (2) and (3) that $[a,c] \notin \mathcal{B}$, and hence by (3) again, $[c,b] \in \mathcal{B}$. Therefore $M \vDash \varphi_n(c,b)$ as required. □

We would also like to isolate the conditions that an indicated class of cuts needs to satisfy.

Definition 2.4. A class $\mathcal{Z} \subseteq \mathcal{C}$ is a *species of cuts* (*species* for short) if

(0) $\mathcal{Z}$ is nonempty;
(1) $\mathcal{Z}$ is invariant under the action of $\mathrm{Aut}(M)$; and
(2) for every $B \in M$, there exists a recursive Σ_1 type $p(x,y)$ over M such that

$$\forall a, b < B\ \left(\exists I \in \mathcal{Z}\ (a \in I < b) \Leftrightarrow M \vDash \bigwedge\!\!\!\bigwedge p(a,b)\right).$$

If I is an element of $\mathcal{Z} \subseteq \mathcal{C}$, then we say that I is a $\mathcal{Z}$*-cut*. Each species of cuts $\mathcal{Z}$ comes equipped with a natural linear order, namely the subset relation, $\subseteq$.

Neighbourhood systems and species of cuts naturally arise from indicators $Y\colon M \times M \to M$ in the sense of GCMA. More generally, this Y might be a class in the sense of M, i.e., segments of Y are parametrically definable in M. Still more generally, our notion of indicator may not in fact be a function at all, but is formed from a family of M-finite functions $Y_B\colon M_{<B} \times M_{<B} \to M$ for various $B \in M$ such that for $B_1 < B_2$ the functions Y_{B_1} and Y_{B_2} agree for all $x, y < B_1$ in the sense that $Y_{B_1}(x,y) > \mathbb{N}$ if and only if $Y_{B_2}(x,y) > \mathbb{N}$.

Definition 2.5. Let $B \in M$, and let $Y\colon M_{<B} \times M_{<B} \to M$ be definable.

- The function Y is said to *indicate* a neighbourhood system $\mathcal{B}$ below B if

$$\forall a, b < B\ ([a,b] \in \mathcal{B} \Leftrightarrow Y(a,b) > \mathbb{N}).$$

- The function Y is said to *indicate* a species of cuts $\mathcal{Z}$ below B if

$$\forall a, b < B\ (\exists I \in \mathcal{Z}\ a \in I < b \Leftrightarrow Y(a,b) > \mathbb{N}).$$

- We say that Y is *monotone* if

$$\forall a, b, u, v \leqslant B\ \Big(a \leqslant u \wedge v \leqslant b \Rightarrow Y(a,b) \geqslant Y(u,v)\Big).$$

Proposition 2.6. *(a) Relative to the other axioms for a neighbourhood system, axiom (5) is equivalent to any of the statements that for all $B \in M$ the neighbourhood system below B is indicated by: a definable function; a monotone definable function; or a recursive type of bounded complexity.*

(b) Relative to the other axioms for a species of cuts, axiom (2) is equivalent to any of the statements that for all $B \in M$ the species is indicated below B by: a definable function; a monotone definable function; or a recursive type of bounded complexity.

Proof. (Sketch.) Fix $\mathcal{B} \subseteq \mathcal{S}$. Pick a recursive Σ_m type $p(x, y)$ such that

$$\forall a, b < B\ \Big([a,b] \in \mathcal{B} \Leftrightarrow M \vDash \bigwedge\!\!\!\bigwedge p(a,b)\Big).$$

Let $\bar{d} \in M$ be the parameters that appear in $p(x, y)$, and write $p(x, y)$ as $p(x, y, \bar{d})$. Then $p(x, y, \bar{d})$ is coded in M by c, say, so that

$$\{(c)_n : n \in \mathbb{N}\} = \{\ulcorner\varphi(x, y, \bar{z})\urcorner : \varphi(x, y, \bar{d}) \in p(x, y, \bar{d})\}.$$

Define a function $Y : M_{<B} \times M_{<B} \to M$ by

$$Y(x, y) = (\mu n)\Big(\neg\, \mathrm{Sat}_{\Sigma_m}((c)_n, [x, y, \bar{d}])\Big)$$

for all $x, y < B$. This is a definable function that indicates $\mathcal{B}$ below B. To obtain a monotone indicator function replace Y with

$$Y'(x, y) = \max\{Y(a, b) : a, b \in [x, y]\}.$$

Since Y and Y' are definable with domain $M_{<B} \times M_{<B}$ they are M-finite. So, they can be coded as a sequence of values by some $y \in M$, say. Then the type

$$p(u, v) = \{Y(u, v) > n : n \in \mathbb{N}\},$$

is a recursive Σ_1 type indicating $\mathcal{B}$ using the parameter y.

The argument for species is similar. □

Every neighbourhood system $\mathcal{B}$ gives rise to a 'largest' species of cuts that it indicates. Similarly, every species of cuts $\mathcal{Z}$ has a natural neighbourhood system that describes it. How to go from a neighbourhood system to a species of cuts and back again is defined next.

Definition 2.7. Given a neighbourhood system $\mathcal{B}$, define $\mathcal{Z}(\mathcal{B})$, *the species of cuts associated with* $\mathcal{B}$, by

$$\mathcal{Z}(\mathcal{B}) = \{I \in \mathcal{C} : \forall [a, b] \in \mathcal{S}\ (I \in [a, b] \Rightarrow [a, b] \in \mathcal{B})\}.$$

Definition 2.8. Given a species of cuts $\mathcal{Z}$, define $\mathcal{B}(\mathcal{Z})$, *the neighbourhood system associated with* $\mathcal{Z}$, by

$$\mathcal{B}(\mathcal{Z}) = \{[a,b] \in \mathcal{S} : \exists I \in \mathcal{Z}\ I \in [a,b]\}.$$

Proposition 2.9. *(a) If $\mathcal{B}$ is a neighbourhood system then $\mathcal{Z}(\mathcal{B})$ is a species of cuts, and if $\mathcal{Z}$ is a species of cuts then $\mathcal{B}(\mathcal{Z})$ is a neighbourhood system.*
(b) If $\mathcal{B}$ is a neighbourhood system, then $\mathcal{B}(\mathcal{Z}(\mathcal{B})) = \mathcal{B}$.
(c) If $\mathcal{Z}$ is a species of cuts, then $\mathcal{Z}(\mathcal{B}(\mathcal{Z})) \supseteq \mathcal{Z}$.

Proof. Straightforward applications of the axioms. □

Given a neighbourhood system $\mathcal{B}$ and $[a,b] \in \mathcal{B}$, there are many cuts $I \in \mathcal{Z}(\mathcal{B})$ with $a \in I < b$. In particular the next definition provides some natural examples. For this definition, recall that, for a nonempty set $A \subseteq M$, $\inf A$ is the greatest initial part of M that is disjoint with A and $\sup A$ is the least initial part of M containing A.

Definition 2.10. Given a neighbourhood system $\mathcal{B}$ and $a, b \in M \cup \{\infty\}$, let

- $M_{\mathcal{B}}(a) = \inf\{c \in M : [a,c] \in \mathcal{B}\}$, and
- $M_{\mathcal{B}}[b] = \sup\{d \in M : [d,b] \in \mathcal{B}\}$.

The notation $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ hides the fact that these may not be defined for all a, b. We say that $M_{\mathcal{B}}(a)$ *exists* if

$$\exists y \in M \cup \{\infty\}\ [a,y] \in \mathcal{B}.$$

Similarly, $M_{\mathcal{B}}[b]$ *exists* if

$$\exists x \in M\ [x,b] \in \mathcal{B}.$$

It is simple to check from the axioms that given $[\![a,b]\!]$ in a neighbourhood system $\mathcal{B}$, both $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ exist and are between a and b. The cuts $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ are respectively the smallest cut in $\mathcal{Z}(\mathcal{B})$ containing a and the largest cut in $\mathcal{Z}(\mathcal{B})$ not containing b. It follows from Proposition 2.3 that $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ must be distinct.

It is time to see some examples.

Example 2.11. The set $\mathcal{B}^{\mathcal{C}} = \{[a,b] \in \mathcal{S} : \forall n \in \mathbb{N}\ a + n < b\}$ is easily seen to be a neighbourhood system. The associated species of cuts is $\mathcal{C}$, the set of all cuts of M.

Example 2.12. Let Y be an indicator in the sense of GCMA and suppose

$$M \vDash \exists x \exists y\, Y(x,y) \geqslant n$$

for every $n \in \mathbb{N}$ to avoid triviality. We call indicators in this old sense *GCMA indicators* in this paper. Set

$$\mathcal{B}^Y = \{[a,b] \in \mathcal{S} : Y(a,b) > \mathbb{N}\} \cup \{[a,\infty] \in \mathcal{S} : \exists b \in M\, Y(a,b) > \mathbb{N}\}.$$

Then $\mathcal{B}^Y$ is a neighbourhood system. The associated species of cuts is $\mathcal{Z}^Y = \mathcal{Z}(\mathcal{B}^Y)$, which is the largest set of cuts indicated by Y. For example, if Y is the Paris–Harrington indicator for cuts satisfying PA, then $\mathcal{Z}^Y$ is the topological closure of the set of cuts satisfying PA; or alternatively, it is the set of all cuts satisfying the Π_2 consequences of PA.

Example 2.13. Recall that M is *short recursively saturated* if each recursive type $p(x)$ that contains a formula of the form $x < a$, where a is a parameter from M, is realised in M. Suppose M is short recursively saturated. Fix a recursive sequence $(t_n(x))_{n\in\mathbb{N}}$ of $\mathscr{L}_{\mathrm{A}}$ Skolem functions with the following properties:

- $\forall n \in \mathbb{N}\, \forall x \in M\ (t_n(x) < t_{n+1}(x))$;
- $\forall n \in \mathbb{N}\, \forall x \in M\ (x < t_n(x) \leqslant t_n(x+1))$; and
- for each $\mathscr{L}_{\mathrm{A}}$ Skolem function $s(x)$ there is an $n \in \mathbb{N}$ such that for all $x \in M$, we have $s(x) < t_n(x)$.

Using short recursive saturation, one can show that the set

$$\mathcal{B}^{\mathrm{elem}} = \{[a,b] \in \mathcal{S} : \forall n \in \mathbb{N}\ (t_n(a) < b)\}$$

is a neighbourhood system. Intervals in $\mathcal{B}^{\mathrm{elem}}$ will be called *elementary intervals*. The corresponding species of cuts, $\mathcal{Z}^{\mathrm{elem}} = \mathcal{Z}(\mathcal{B}^{\mathrm{elem}})$, is the species of elementary cuts of M. By a diagonalisation argument, it can be seen there is no definable function $Y\colon M^2 \to M$ such that

$$Y(a,b) > \mathbb{N} \Leftrightarrow [a,b] \in \mathcal{B}^{\mathrm{elem}}$$

for all $a, b \in M$. Therefore, our definition of a neighbourhood system is strictly more general than its counterpart in GCMA.

For $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$ the cuts $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ are familiar cuts, usually denoted $M(a)$ and $M[b]$.

In certain circumstances, the neighbourhood system $\mathcal{B}^{\mathrm{elem}}$ can be regarded as the 'finest' such system, as the following proposition shows.

Proposition 2.14. *Suppose M is recursively saturated and let $\mathcal{B}$ be a neighbourhood system such that for each $a \in M$ there is $c \in M$ with $[a,c] \in \mathcal{B}$. Then $\mathcal{B} \supseteq \mathcal{B}^{\mathrm{elem}}$.*

Proof. For each $a \in M$, the semi-interval $[a,\infty]$ is in $\mathcal{B}$ since there is some $c \in M$ with $[a,\infty] \supseteq [a,c] \in \mathcal{B}$. Now let $[a,b] \in \mathcal{B}^{\mathrm{elem}}$ and $c \in M$ with

$[a,c] \in \mathcal{B}$. Assume $b \neq \infty$, or else we are done. Note that $b > \mathrm{cl}(a)$, and so by recursive saturation, there is an automorphism g of M fixing a such that $c^g < b$. It follows from the axioms that $[a,b] \in \mathcal{B}$. □

It can easily be checked that some facts about indicators transfer to this more general setting. The following lemma is formulated in terms of the standard cut because the region around $\mathbb{N}$ is the place where we are mostly interested in. It is also true of other cuts, as we leave the reader to verify.

Lemma 2.15. *Let $\mathcal{B}$ be a neighbourhood system, let $B \in M$, and let Y be an indicator for $\mathcal{B}$ below B. If $[\![a,b]\!] \subseteq M_{<B}$ is a $\mathcal{B}$-interval, then*

$$\{n > \mathbb{N} : M \models \exists [u,v] \subseteq [\![a,b]\!]\ (Y(u,v) = n)\} \subseteq_{\mathrm{dcf}} M \setminus \mathbb{N}.$$

Proof. Take a $\mathcal{B}$-interval $[\![a,b]\!] \subseteq M_{<B}$ and define X to be the set

$$\{n \in \mathbb{N} : \exists [u,v] \subseteq [\![a,b]\!]\ (Y(u,v) = n)\}.$$

Note that X is nonempty.

Suppose $X \not\subseteq_{\mathrm{cf}} \mathbb{N}$. Then X has an upper bound in $\mathbb{N}$, say D. Now for every $x \in [\![a,b]\!]$,

$$[x,b] \in \mathcal{B} \text{ iff } Y(x,b) > \mathbb{N} \text{ iff } Y(x,b) > D.$$

Therefore, since the set $\{x \in [\![a,b]\!] : [x,b] \in \mathcal{B}\}$ contains a and is bounded above by b, it has a maximum element, say $x^* \in M$. So $[x^*,b] \in \mathcal{B}$ but $[x^*+1,b] \notin \mathcal{B}$. This contradicts (2) and (3) in the definition of a neighbourhood system. □

3 The topology on $\mathcal{Z}$ and enforceable properties

Kotlarski seems to be the first person who explicitly studied families of cuts with their topology obtained from the order relation. His paper, *Some remarks on initial segments in models of Peano arithmetic* [12], is of particular relevance here. In the terminology here, his Theorem 1 shows that if Y is the Paris–Harrington indicator then $\mathcal{Z}^Y$ is homeomorphic to the Cantor set, and the species $\mathcal{Z}_{\mathrm{PA}}$ of cuts satisfying PA is a dense subset of $\mathcal{Z}^Y$. His Theorem 3 (attributed to Paris) shows furthermore that $\mathcal{Z}_{\mathrm{PA}}$ is meagre in $\mathcal{Z}^Y$. See also earlier papers by Kotlarski [10,11] which investigate the species of elementary cuts, and the appendix to Smoryński [15] which summarises this work of Kotlarski.

The class of all cuts $\mathcal{C}$ is linearly ordered by inclusion. Every linear order carries a natural topology, the so-called order topology, in which the basic open sets are open intervals. A species of cuts $\mathcal{Z}$ can therefore be considered as a topological space, where the topology on $\mathcal{Z}$ is the subspace topology inherited from $\mathcal{C}$. Each $\mathcal{B}(\mathcal{Z})$-interval $[\![a, b]\!]$ determines an open set

$$\{I \in \mathcal{Z} : M_{\mathcal{B}(\mathcal{Z})}(a) < I < M_{\mathcal{B}(\mathcal{Z})}[b]\}$$

in $\mathcal{Z}$, and open sets of this form generate the topology on $\mathcal{Z}$.

Proposition 3.1. *Given a species of cuts $\mathcal{Z}$, the topological closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ in $\mathcal{C}$ is $\mathcal{Z}(\mathcal{B}(\mathcal{Z}))$.*

Proof. If $I \notin \overline{\mathcal{Z}}$, then there is $[a, b] \in \mathcal{B}(\mathcal{C})$ containing I such that no $\mathcal{Z}$-cut is between a and b. However, this means $[a, b] \notin \mathcal{B}(\mathcal{Z})$ and hence $I \notin \mathcal{Z}(\mathcal{B}(\mathcal{Z}))$. Conversely if $I \in \overline{\mathcal{Z}}$ then every $[a, b] \in \mathcal{B}(\mathcal{C})$ which contains I also contains some $J \in \mathcal{Z}$. Therefore $I \in \mathcal{Z}(\mathcal{B}(\mathcal{Z}))$. □

Paris and Kirby call two families of cuts *symbiotic* if they have the same indicators. This generalises immediately to our context, explaining perhaps our use of the word 'species'.

Definition 3.2. Two species of cuts $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are *symbiotic* if every open set containing a cut from one species contains a cut from the other, i.e., if their closures are equal: $\overline{\mathcal{Z}_1} = \overline{\mathcal{Z}_2}$.

In the rest of this section, we will assume M *is countable,* $\mathcal{B}$ *is a neighbourhood system, and* $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$ *is the closed species of cuts associated with* $\mathcal{B}$.

Proposition 3.3. *The space $\mathcal{Z}$ is order-isomorphic (and hence homeomorphic) to either*

(i) 2^ω*, the Cantor set with its usual ordering and topology; or*
(ii) $2^\omega + 1$*, the Cantor set with an additional isolated point greater than all the others.*

Proof. Fix an enumeration $(x_n)_{n\in\mathbb{N}}$ of M.

Define the sequence $([\![a_\sigma, b_\sigma]\!])_{\sigma\in 2^{<\omega}}$ of $\mathcal{B}$-intervals recursively as follows.

Let $a_\varnothing = 0$. By axioms (0) and (4) for a neighbourhood system, $0 \ll \infty$. If there is $b \in M$ such that $[b, \infty] \notin \mathcal{B}$, then choose such $b_\varnothing$ to be b; otherwise, define $b_\varnothing = \infty$.

Let $n \in \mathbb{N}$ and $\sigma \in 2^n$ such that $[\![a_\sigma, b_\sigma]\!]$ is defined. If there is $c_\sigma \in M$ such that $a_\sigma \ll c_\sigma \ll b_\sigma$, then define

$$[\![a_{\sigma 0}, b_{\sigma 0}]\!] = \begin{cases} [\![a_\sigma, x_n]\!], & \text{if } a_\sigma \ll x_n \ll b_\sigma; \\ [\![x_n, c_\sigma]\!], & \text{if } a_\sigma \leqslant x_n \text{ and } [a_\sigma, x_n] \notin \mathcal{B}; \\ [\![a_\sigma, c_\sigma]\!], & \text{otherwise;} \end{cases}$$

and

$$[\![a_{\sigma 1}, b_{\sigma 1}]\!] = \begin{cases} [\![x_n, b_\sigma]\!], & \text{if } a_\sigma \ll x_n \ll b_\sigma; \\ [\![c_\sigma, x_n]\!], & \text{if } x_n \leqslant b_\sigma \text{ and } [x_n, b_\sigma] \notin \mathcal{B}; \\ [\![c_\sigma, b_\sigma]\!], & \text{otherwise.} \end{cases}$$

If there is no $c \in M$ satisfying $a_\sigma \ll c \ll b_\sigma$, then set both $[\![a_{\sigma 0}, b_{\sigma 0}]\!]$ and $[\![a_{\sigma 1}, b_{\sigma 1}]\!]$ to be $[\![a_\sigma, b_\sigma]\!]$.

Using the axioms and the enumeration of M, it is straightforward to check that every non-isolated cut in $\mathcal{Z}$ is the limit of an increasing sequence $(a_{\varepsilon\restriction n})_{n\in\omega}$ for some $\varepsilon\colon \omega \to 2$, and conversely any such limit is a cut in $\mathcal{Z}$. We omit the details.

The case when $\mathcal{Z}$ turns out to be order-isomorphic to $2^\omega + 1$ is when no $c \in M$ can be found such that $a_\sigma \ll c \ll b_\sigma$ for some $\sigma \in 2^{<\omega}$. Note that in this case, $b_\sigma = \infty$ by Proposition 2.3, and so we must have $b_\sigma = b_\varnothing = \infty$. It follows that $[a_\sigma, c] \notin \mathcal{B}$ for all $c \in M$ greater than a_σ, because otherwise, either $[c, \infty] \in \mathcal{B}$ or it is not. The former contradicts the fact that we are in this case, while the latter contradicts our choice that $b_\varnothing = \infty$. Therefore, $[c, \infty] \in \mathcal{B}$ for all sufficiently large $c \in M$ by axiom (3). All such $[\![c, \infty]\!]$ can only contain one cut, and so it has to be M. This is the isolated greatest element of $\mathcal{Z}$. □

The various cases implicit in the proof just given do all occur.

Example 3.4. Let Y be the Paris–Harrington indicator for initial segments satisfying PA. Then $\mathcal{Z} = \mathcal{Z}^Y$ is closed by Example 2.12, and it is order-isomorphic to 2^ω. There are proper cuts in $\mathcal{Z}$ arbitrarily high in M and also nonstandard cuts in $\mathcal{Z}$ arbitrarily low in M, as well as both end points, M and $\mathbb{N}$ in $\mathcal{Z}$.

Example 3.5. Suppose $M \vDash \neg\mathrm{Con}(\mathrm{PA})$. Let Y be an indicator for initial segments satisfying $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$. Then once again $\mathcal{Z} = \mathcal{Z}^Y$ is closed and is order-isomorphic to 2^ω, but this time there is some $B \in M$ above all $I \in \mathcal{Z}$.

Example 3.6. Suppose M is *short*, i.e., $M = \overline{\mathrm{cl}}(a)$ for some $a \in M$, or equivalently, M has no proper elementary initial segment containing a. Suppose further that M is short recursively saturated. Then $\mathcal{Z} = \mathcal{Z}^{\mathrm{elem}}$ is a closed species by Example 2.13. The full model M itself is clearly in $\mathcal{Z}$, but $\mathcal{Z}$ does not have arbitrarily large proper cuts of M, since if $a \in I \prec_{\mathrm{e}} M$ then $I = M$. So in this case $\mathcal{Z} \cong 2^\omega + 1$.

Proposition 3.3 makes a whole range of topological tools available to us. For example, we now know that $\mathcal{Z}$, as a topological space, is compact, totally disconnected, of cardinality $2^{\aleph_0}$, and homeomorphic to a complete metric space. It is perfect if and only if M is not an isolated point. In addition, the Baire Category Theorem applies. Recall a set is *comeagre* if it contains a countable intersection of dense open sets.

Baire Category Theorem. A comeagre subset in a complete metric space is dense in this space.

In particular, comeagre sets in a complete metric space are nonempty. In fact, using a tree argument, one can show that every comeagre set in our space $\mathcal{Z}$ has size the continuum. The intersection of countably many comeagre sets is comeagre, and the set $\mathcal{Z} \setminus \{I\}$ is comeagre for any non-isolated point $I \in \mathcal{Z}$. Hence the complement of any countable set of non-isolated points is comeagre.

Dense subsets of a complete species are exactly those that are *indicated* in the sense of Kirby–Paris [5]. This is one point of interest in comeagre sets of cuts. Comeagre sets have many nice properties, including a useful game-theoretic characterisation.

Definition 3.7. The *Banach–Mazur game on* $\mathcal{B}$ is the following game.

- There are two players, called $\forall$ and $\exists$.
- Starting with $\forall$, the two players alternatingly choose a $\mathcal{B}$-interval that is a subinterval of the previously chosen one.
- The game terminates in ω many steps.

A play of this game gives rise to a sequence $(\llbracket a_n, b_n \rrbracket)_{n\in\mathbb{N}}$. The cut $\sup\{a_n : n \in \mathbb{N}\}$ is called the *outcome* of the play. The player $\exists$ can always play in such a way to ensure that this is a cut lying in $\mathcal{Z}$.

A property P of cuts is *enforceable* if $\exists$ has a way to ensure the outcome of a play has property P. Similarly, a subset $\mathcal{P}$ of $\mathcal{Z}$ is *enforceable* if the property of being an element of $\mathcal{P}$ is enforceable.

By 'dovetailing' several strategies together, it is easy to see that $\exists$ can play to enforce countably many properties simultaneously, provided she can enforce each individual one. This observation is part of the proof of Banach's characterisation of comeagre sets.

Theorem 3.8 (Banach)**.** *A subset $\mathcal{P} \subseteq \mathcal{Z}$ is enforceable if and only if it is comeagre in $\mathcal{Z}$.*

From the point of view of Baire category, an enforceable property P of $\mathcal{Z}$-cuts is satisfied by a large set of cuts $I \in \mathcal{Z}$. So a 'general' (i.e., not carefully chosen or exceptional) example of a cut in $\mathcal{Z}$ would be expected to have many such enforceable properties. It cannot satisfy all of them unless I is actually isolated in $\mathcal{Z}$ because $\mathcal{Z} \setminus \{I\}$ is comeagre. A *generic cut* I in $\mathcal{Z}$ is one that satisfies as many enforceable properties as is reasonably possible. Say that $\mathcal{P} \subseteq \mathcal{Z}$ is *invariant under automorphisms of* M if $\{I^g : I \in \mathcal{P}\} = \mathcal{P}$ for each

$g \in \mathrm{Aut}(M)$.

Definition 3.9. We say that a $\mathcal{Z}$-cut I is *generic (in $\mathcal{Z}$)* or *$\mathcal{Z}$-generic* if I is an element of each comeagre $\mathcal{P} \subseteq \mathcal{Z}$ invariant under automorphisms of M.

For a simple example when generic cuts might exist, suppose there is some cut $I \in \mathcal{Z}$ such that the set

$$\{I^g \in \mathcal{Z} : g \in \mathrm{Aut}(M)\}$$

is comeagre. Then the cut I is generic. To see this, let P be an invariant enforceable property and play the Banach–Mazur game to enforce P and the property of being conjugate to I simultaneously. The resulting cut has both of these properties, and hence I satisfies P. The next result gives a more useful generalisation of this observation.

Theorem 3.10. *Let $M \models \mathrm{PA}$ be countable and $\mathcal{Z}$ a closed species of cuts order-isomorphic to 2^ω. Suppose $\mathcal{G}$ is a set of $\mathcal{Z}$-cuts such that*

(i) $\mathcal{G}$ is a dense subset of $\mathcal{Z}$ that is invariant under automorphisms of M; and
(ii) for all $I \in \mathcal{G}$ and all $\bar{c} \in M$, there is an interval $[\![a, b]\!] \in \mathcal{B}(\mathcal{Z})$ containing I in which all cuts in $\mathcal{G}$ are conjugate over $\bar{c}$.

Then $\mathcal{G}$ is a comeagre set of cuts in $\mathcal{Z}$ and the cuts in $\mathcal{G}$ are precisely the $\mathcal{Z}$-generic cuts.

Proof. We start by showing that the property of being in $\mathcal{G}$ is enforceable. This will show that $\mathcal{G}$ contains all generic cuts. We play the Banach–Mazur game. Fix an enumeration $(x_n)_{n \in \mathbb{N}}$ of M. At stage n in the game we will have chosen

- $I_0, I_1, \ldots, I_{n-1} \in \mathcal{G}$;
- $g_1, g_2, \ldots, g_{n-1} \in \mathrm{Aut}(M)$;
- $c_0, c_1, \ldots, c_{n-1} \in M$; and
- a descending sequence $[\![a_0, b_0]\!], [\![a_1, b_1]\!], \ldots, [\![a_{n-1}, b_{n-1}]\!]$ of $\mathcal{B}(\mathcal{Z})$-intervals

such that for all $i < n$,

- $I_i \in [\![a_i, b_i]\!]$;
- $g_i \in \mathrm{Aut}(M, c_0, c_1, \ldots, c_{i-1})$ with $I_{i-1}^{g_i} = I_i$; and
- all $\mathcal{G}$-cuts in $[\![a_i, b_i]\!]$ are conjugate over $c_0, c_1, \ldots, c_i$.

The intervals $[\![a_i, b_i]\!]$ are our plays in the game.

Given our opponent's move $[\![u, v]\!]$ in the game, we first choose a $\mathcal{G}$-cut $I_n \in [\![u, v]\!]$ using the density of $\mathcal{G}$. If $n > 0$, then we will also need to choose an automorphism $g_n \in \mathrm{Aut}(M, c_0, c_1 \ldots, c_{n-1})$ such that $I_{n-1}^{g_n} = I_n$. This can be done using the inductive conditions since $I_{n-1}, I_n \in [\![a_{n-1}, b_{n-1}]\!]$. If n is even, $n = 2k$ say, then we set

$$c_n = c_{2k} = x_k.$$

If n is odd, $n = 2k + 1$ say, then we choose

$$c_n = c_{2k+1} = x_k^{g_1 g_2 \cdots g_{2k+1}}$$

instead. Using condition (ii) on $\mathcal{G}$, choose $[\![a_n, b_n]\!] \subseteq [\![u, v]\!]$ containing I_n in which all $\mathcal{G}$-cuts are conjugate over $c_0, c_1, \ldots, c_n$. We play the interval $[\![a_n, b_n]\!]$ in the game.

The play continues in this fashion and constructs a cut $J \in \mathcal{Z}$ which is the limit of $(a_n)_{n\in\mathbb{N}}$. We must show that $J \in \mathcal{G}$. In view of condition (i), it suffices to show that I_0 and J are conjugate.

Observe that g_{n+i+1} fixes c_n for each $n, i \in \mathbb{N}$. So if $x = x_k$, then $c_{2k+1} = x_k^{g_1 g_2 \cdots g_{2k+1}}$ is fixed by g_{2k+2}, g_{2k+3}, etc. Therefore, for each $x \in M$, there is $k \in \mathbb{N}$ such that

$$x^{g_1 g_2 \cdots g_k} = x^{g_1 g_2 \cdots g_{k+1}} = x^{g_1 g_2 \cdots g_{k+2}} = \cdots$$

We define $g\colon M \to M$ so that each $x \in M$ is mapped to the eventual value of $(x^{g_1 g_2 \cdots g_l})_{l\in\mathbb{N}}$. It is easy to see that g preserves the $\mathscr{L}_\mathrm{A}$ structure and is injective. It is surjective because for each $y \in M$, there is $k \in \mathbb{N}$ such that $y = x_k = c_{2k}$, and so

$$g\colon y^{g_{2k}^{-1} g_{2k-1}^{-1} \cdots g_1^{-1}} \mapsto y.$$

Finally, $I_0^g = J$ because g is the 'limit' of $(g_1 g_2 \cdots g_l)_{l\in\mathbb{N}}$ while J is the 'limit' of $(I_0^{g_1 g_2 \cdots g_l})_{l\in\mathbb{N}}$. This completes the proof that $\mathcal{G}$ is enforceable and every generic cut is in $\mathcal{G}$.

To show that every $I \in \mathcal{G}$ is generic, let $\mathcal{P} \subseteq \mathcal{Z}$ be $\mathrm{Aut}(M)$-invariant and comeagre, and $[\![a, b]\!]$ be chosen so that $I \in [\![a, b]\!]$ and every $\mathcal{G}$-cut in $[\![a, b]\!]$ is conjugate to I. Then we play the Banach–Mazur game starting with $[\![a, b]\!]$ enforcing $\mathcal{P}$ and $\mathcal{G}$ simultaneously to construct some $J \in \mathcal{G} \cap \mathcal{P}$ with $J \in [\![a, b]\!]$. Then I is conjugate to J and hence satisfies $\mathcal{P}$, as required. □

Question 3.11. Suppose the set $\mathcal{G}$ of $\mathcal{Z}$-generic cuts is comeagre in $\mathcal{Z}$. Does it follow that conditions (i) and (ii) in the statement of Theorem 3.10 hold?

4 Examples of enforceable properties of cuts

The objective of this section is to extend the results of enforceability of various properties of cuts in GCMA to the setting of this paper. We make the global assumption that *our model $M \vDash \mathrm{PA}$ is countable and nonstandard, and $\mathscr{Z}$ is a closed species of cuts order-isomorphic to* 2^ω. We let $\mathcal{B} = \mathcal{B}(\mathscr{Z})$ be the associated neighbourhood system. To apply the results under these assumptions when $\mathscr{Z} \cong 2^\omega + 1$, we can replace $\mathscr{Z}$ with $\mathscr{Z}_0 = \mathscr{Z} \setminus \{M\}$, which is also closed.

Proposition 4.1. *It is enforceable that a $\mathscr{Z}$-cut is not an ω-limit.*

Proof. By assumption, no $I \in \mathscr{Z}$ is isolated so $\mathscr{Z} \setminus \{I\}$ is comeagre. The proposition follows from the countability of M as there are countably many cuts which are ω-limits. □

Proposition 4.2. *It is enforceable that*

$$I \neq M_{\mathcal{B}}(a) \text{ and } I \neq M_{\mathcal{B}}[a] \text{ whenever } a \in M$$

for a $\mathscr{Z}$-cut I.

Proof. There are countably many cuts of the form $M_{\mathcal{B}}(a)$ or $M_{\mathcal{B}}[a]$. □

In a similar way one can see that it is enforceable that a cut is not definable over finitely many parameters from M in any reasonable logic, such as infinitary logic or second order logic, since there are only countably many conjugates of these parameters.

Definition 4.3. Let $\mathscr{L}_\mathrm{A}^I$ denote the language obtained from $\mathscr{L}_\mathrm{A}$ by adding an extra unary relation symbol, which will usually represent a cut of M. The language obtained from $\mathscr{L}_\mathrm{A}^I$ by adding all $\mathscr{L}_\mathrm{A}$ Skolem functions is denoted by $\mathscr{L}_\mathrm{Sk}^I$.

Definition 4.4. A $\Pi_2^{(I)}$ *formula* is an $\mathscr{L}_\mathrm{A}^I$ formula of the form

$$\forall \bar{x} \in I \; \exists \bar{y} \in I \; \theta(\bar{x}, \bar{y}, \bar{z})$$

where $\theta(\bar{x}, \bar{y}, \bar{z}) \in \Delta_1$.

Proposition 4.5. *It is enforceable that a $\mathscr{Z}$-cut I has the property that $\mathbb{N}$ is $\Pi_2^{(I)}$ definable with parameters in (M, I) for a $\mathscr{Z}$-cut I.*

Proof. We play a Banach–Mazur game on $\mathcal{B}$. Suppose $\forall$ plays $[\![a, b]\!]$ in his first move, and without loss of generality we may assume b is finite. Let $Y \in M$ be a monotone indicator for $\mathcal{B}$ below $b+1$. We show that $\exists$ can force the outcome of the play I to satisfy

$$\{n \in M : M \vDash \forall x \in I \; \exists y \in I \; Y(x, y) \geqslant n\} = \mathbb{N}.$$

Note that, since we can make the outcome to be a $\mathscr{Z}$-cut, it is clear that we can enforce I to satisfy $\{n \in M : M \vDash \forall x \in I \; \exists y \in I \; Y(x, y) \geqslant n\} \supseteq \mathbb{N}$. Let

$n \in M$ be nonstandard, and suppose that $\exists$ is given $[\![u, v]\!] \subseteq [\![a, b]\!]$ to play in. Using Lemma 2.15, let $[\![x_n, y_n]\!] \subseteq [\![u, v]\!]$ such that $Y(x_n, y_n) < n$. Using the countability of M, player $\exists$ can do this for every nonstandard $n \in M$ in any single play. Now, if I is an outcome of this play and $n \in M$ is nonstandard, then we have $x_n \in I < y_n$ such that

$$Y(x_n, y) \leqslant Y(x_n, y_n) < n$$

for each $y \in I$ by the monotonicity of Y. This proves the claim. □

Remark. In the terminology of Kirby [4, Definition 4.5], the above proof shows that one can enforce the *index* of a cut corresponding to an indicator to be $\mathbb{N}$.

Corollary 4.6. *It is enforceable that a $\mathcal{Z}$-cut I has the property that (M, I) is not $\Pi_2^{(I)}$ recursively saturated.*

Proof. If not, apply $\Pi_2^{(I)}$ recursive saturation to the set of formulas $\{z > n : n \in \mathbb{N}\} \cup \{\forall x \in I\ \exists y \in I\ \theta(x, y, z, \bar{a})\}$ where $\theta(x, y, z, \bar{a})$ is the Δ_1 formula from the last proposition. □

Enforceability results related to the Kirby–Paris notions of semiregularity and regularity are proved in GCMA. A slight modification of the Grzegorczyk hierarchy as used there gives us the following.

Definition 4.7. The neighbourhood system $\mathcal{B}$ is said to be *relatively indestructible* if for every $[\![a, b]\!] \in \mathcal{B}$, there is an element $c \in M$ such that

$$a = (c)_0 \ll (c)_1 \ll \cdots \ll (c)_{a+1} = b.$$

Using the same ideas it is straightforward to modify the combinatorial arguments given as Theorem 4.13 and Theorem 4.15 in GCMA to obtain the following results showing that semiregularity is the best one can hope for in the sense of the 'classical' Paris–Kirby hierarchy of combinatorial properties.

Proposition 4.8. *Semiregularity is enforceable if and only if $\mathcal{B}$ is relatively indestructible.*

Proposition 4.9. *The property of being not regular is enforceable.*

5 Pregenerics and the existence of generic cuts

Throughout this section, we work with a recursive enumeration $(\theta_i(x, y, \bar{z}))_{i \in \mathbb{N}}$ of $\mathscr{L}_{\mathrm{A}}$ formulas in the free variables $x, y, \bar{z}$. We fix a neighbourhood system $\mathcal{B}$, and its associated closed species $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$. We continue the global assumption of the last section that $\mathcal{Z}$ has no isolated point.

Our objective is to prove results showing the existence of generic cuts. Our motivation is Theorem 3.10 and the problem we address is to identify those intervals which are sufficiently homogeneous for many cuts in them to be conjugate. The existence of generic cuts relative to an indicator Y was shown in GCMA by a related 'self-similarity' property of intervals, that of being 'constant', together with a 'smallness' notion. We give the first of these definitions here.

Definition 5.1. Let $\bar{c} \in M$. An interval $[\![a,b]\!] \in \mathcal{B}$ is *constant over* $\bar{c}$ *(with respect to* $\mathcal{B}$*)* if

$$\forall x \in [\![a,b]\!]\ \forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x' \in [\![u,v]\!]\ \ \mathrm{tp}(x,\bar{c}) = \mathrm{tp}(x',\bar{c}).$$

We shall present a two-variable version of this self-similarity idea, which seems to give a more elegant approach. Intervals having this stronger self-similarity property will be called *pregeneric*, and it will be clear that a pregeneric interval is constant in the sense of GCMA.

It will turn out that, by an argument similar to the one in GCMA, pregeneric intervals exist in abundance in arithmetically saturated models of PA. We shall study this argument much more closely. This investigation will reveal that although arithmetic saturation is essential for the full argument, a large part of the proof goes through without any countability or saturation assumption. For applications to understanding truth in expanded structures of the form (M, I) we will be particularly interested in how the arguments can be adapted to notions of self-similarity with respect to finite sets of formulas. This increases the number of technical details but in other respects the main ideas are straightforward and similar to those in the earlier paper.

Definition 5.2. Let $x, y, x', y', \bar{c} \in M$ and $n \in \mathbb{N}$. We write $(x, y, \bar{c}) \equiv_n (x', y', \bar{c})$ to mean

$$\bigwedge_{i \leqslant n} (\theta_i(x, y, \bar{c}) \leftrightarrow \theta_i(x', y', \bar{c})),$$

and write $(x, y, \bar{c}) \equiv (x', y', \bar{c})$ to mean

$$\bigwedge_{i \in \mathbb{N}} (\theta_i(x, y, \bar{c}) \leftrightarrow \theta_i(x', y', \bar{c})).$$

More generally, for $u_1, u_2, \ldots, u_n, v_1, v_2 \ldots, v_n \in M$, we write $(u_1, \ldots, u_n) \equiv (v_1, \ldots, v_n)$ to mean $\mathrm{tp}(u_1, \ldots, u_n) = \mathrm{tp}(v_1, \ldots, v_n)$.

Definition 5.3. Let $[a, b]$ be a finite semi-interval, $n, k, \bar{c} \in M$, and Y an indicator for $\mathcal{B}$ below $b + 1$. We say that $[a, b]$ is $(n, k)_Y$*-pregeneric over* $\bar{c}$ if

$Y(a,b) \geqslant k$ and for all $x, y \in [a,b]$

$$\forall [u,v] \subseteq [a,b] \, \Big(Y(u,v) \geqslant k \rightarrow \exists x', y' \in [u,v] \, ((x,y,\bar{c}) \equiv_n (x',y',\bar{c})) \Big).$$

We shall omit the subscript Y if the indicator in consideration is clear from context.

To prove the existence of (n,k)-pregeneric intervals, we use the tree argument given in GCMA. The only difference here is that the tree is now finite.

Definition 5.4. Let $[a,b] \in \mathcal{S}$ be finite, and let Y be a monotone indicator for $\mathcal{B}$ below $b+1$. Fix $\bar{c} \in M$. For $i \in \mathbb{N}$, define $e_i \colon M_{\leqslant b} \times M_{\leqslant b} \rightarrow M$ by setting $e_i(r,s)$ to be

$$\max \Big\{ l \in M : \exists [r',s'] \subseteq [r,s] \, \Big(Y(r',s') = l \wedge \forall x,y \in [r',s'] \, \neg\theta_i(x,y,\bar{c}) \Big) \Big\}$$

for each $r, s \leqslant b$. The *tree of possibilities from* $[a,b]$ *over* $\bar{c}$ *(with respect to* Y*)* is a sequence $([r_\sigma, s_\sigma])_{\sigma \in 2^{<\omega}}$ of semi-intervals defined recursively as follows.

- Set $[r_\varnothing, s_\varnothing] = [a,b]$.
- Let $m \in \mathbb{N}$ and $\sigma \in 2^m$ such that $[r_\sigma, s_\sigma]$ is defined. Set $[r_{\sigma 0}, s_{\sigma 0}] = [r_\sigma, s_\sigma]$ and let $[r_{\sigma 1}, s_{\sigma 1}] \subseteq [r_\sigma, s_\sigma]$ be the unique semi-interval such that $r_{\sigma 1}$ is the least r in $[r_\sigma, s_\sigma]$ such that

  $$\exists s \in [r_\sigma, s_\sigma] \, \Big(Y(r,s) \geqslant e_m(r_\sigma, s_\sigma) \wedge \forall x,y \in [r,s] \, \neg\theta_m(x,y,\bar{c}) \Big),$$

  and $s_{\sigma 1}$ is the greatest s in $[r_\sigma, s_\sigma]$ such that

  $$\forall x,y \in [r_{\sigma 1}, s] \, \neg\theta_m(x,y,\bar{c}).$$

Remark. Note that the function e_i defined above is dependent on and uniquely determined by the choice of $\bar{c} \in M$ and the indicator Y. Note also that both e_i and the tree of possibilities are uniformly definable in (M, Sat) for all partial inductive satisfaction class Sat for M. This is also true for $(n,k)_Y$-pregenericity over a tuple $\bar{c} \in M$.

The idea is that given a large enough finite semi-interval $[a,b]$ and a formula $\theta(x,y)$, exactly one of two things has to happen: either there is a large subinterval of $[a,b]$ in which no pair of elements satisfies $\theta(x,y)$, or there is not. In the first case, the witnessing subinterval is homogeneous for $\theta(x,y)$, simply because no pair of elements in there satisfies this formula. In the second case, the whole semi-interval is already homogeneous for $\theta(x,y)$, because by assumption, every large enough subinterval contains a pair of elements satisfying $\theta(x,y)$. In either case, we get a sufficiently large subinterval that is homogeneous for $\theta(x,y)$.

We can repeat this argument with all $\mathscr{L}_{\mathrm{A}}$ formulas. It is sometimes quite hard to find out which case we are in, but we definitely know what possibilities we can have. This gives rise to the tree of possibilities defined above. We do not need to know which way down the tree we have to go. We only need to know there is a way that works.

Lemma 5.5. *Let $[a, b]$ be a finite semi-interval, let Y be a monotone indicator for $\mathcal{B}$ below $b + 1$, and let $\bar{c}, k \in M$ such that $Y(a, b) \geqslant k$. If $([r_\sigma, s_\sigma])_{\sigma \in 2^{<\omega}}$ is the tree of possibilities from $[a, b]$ over $\bar{c}$ then*

$$\forall m \in \mathbb{N}\ \exists! \sigma \in 2^m\ \Big(Y(r_\sigma, s_\sigma) \geqslant k \wedge \forall i < m\ (\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) < k)\Big).$$

Proof. This can be proved by an easy induction on m. □

It is then down to checking how many formulas we need to guarantee a certain amount of pregenericity.

Definition 5.6. Let $\beta\colon \mathbb{N} \to \mathbb{N}$ be the function defined by: for all $n \in \mathbb{N}$, the number $\beta(n)$ is the least $m \in \mathbb{N}$ such that

> if $\varphi(x, y, \bar{z})$ is a Boolean combination of formulas in $\{\theta_i(x, y, \bar{z}) : i \leqslant n\}$, then there is a formula $\varphi'(x, y, \bar{z}) \in \{\theta_i(x, y, \bar{z}) : i \leqslant m\}$ that is logically equivalent to $\varphi(x, y, \bar{z})$.

Theorem 5.7. *Let n be a natural number, $[a, b]$ a finite semi-interval, $k, \bar{c} \in M$, and Y a monotone indicator for $\mathcal{B}$ below $b+1$ such that $Y(a, b) \geqslant k$. Then $[a, b]$ contains a semi-interval that is $(n, k)_Y$-pregeneric over $\bar{c}$. Moreover, if* Sat *is a partial inductive satisfaction class for M, then one such semi-interval is definable in (M, Sat) uniformly in the parameters $a, b, \bar{c}, Y, n, k$.*

Proof. Let $([r_\sigma, s_\sigma])_{\sigma \in 2^{<\omega}}$ be the tree of possibilities from $[a, b]$ over $\bar{c}$. Using Lemma 5.5, define the function $\pi\colon \mathbb{N} \to 2^{<\omega}$ by setting $\pi(m)$ to be the unique $\sigma \in 2^m$ such that

$$Y(r_\sigma, s_\sigma) \geqslant k \wedge \forall i < m\ (\sigma(i + 1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) < k)$$

for each $m \in \mathbb{N}$. It can then be checked that $[r_{\pi(\beta(n))}, s_{\pi(\beta(n))}] \subseteq [a, b]$ is $(n, k)_Y$-pregeneric over $\bar{c}$ for every $n \in \mathbb{N}$.

The 'moreover' part can be proved by a careful check of all the steps, and is left to the reader. □

By noting that almost everything in the above argument is coded in M, one can prove the same statement with fully pregeneric intervals in a similar way.

Definition 5.8. Let $\bar{c} \in M$. An interval $[\![a, b]\!] \in \mathcal{B}$ is *pregeneric over $\bar{c}$ (with respect to $\mathcal{B}$)* if

$$\forall x, y \in [\![a, b]\!]\ \forall [\![u, v]\!] \subseteq [\![a, b]\!]\ \exists x', y' \in [\![u, v]\!]\ (x, y, \bar{c}) \equiv (x', y', \bar{c}).$$

We say that a $\mathcal{B}$-interval is *pregeneric (with respect to $\mathcal{B}$)* if it is pregeneric over 0.

Theorem 5.9. *Suppose M is arithmetically saturated. Let $\bar{c} \in M$. Then every $\mathcal{B}$-interval contains a subinterval pregeneric over $\bar{c}$.*

Proof. Let $\bar{c} \in M$ and $[\![a, b]\!] \in \mathcal{B}$. Without loss of generality, assume $b \neq \infty$. Fix a monotone indicator Y for $\mathcal{B}$ below $b + 1$, and let $([r_\sigma, s_\sigma])_{\sigma \in 2^{<\omega}}$ be the tree of possibilities from $[\![a, b]\!]$ over $\bar{c}$. By recursive saturation, this tree of possibilities and thus $(Y(r_\sigma, s_\sigma))_{\sigma \in 2^{<\omega}}$ are coded in M. Using the strength of $\mathbb{N}$ in M, let $d > \mathbb{N}$ such that

$$\forall \sigma \in 2^{<\omega} \; (Y(r_\sigma, s_\sigma) > d \Leftrightarrow Y(r_\sigma, s_\sigma) > \mathbb{N}).$$

In particular, $Y(a, b) > d$ since $[\![a, b]\!] \in \mathcal{B}$. By Lemma 5.5, we have

$$\forall m \in \mathbb{N} \; \exists! \sigma \in 2^m \; \Big(Y(r_\sigma, s_\sigma) > d \wedge \forall i < m \; (\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) \leqslant d)\Big).$$

Using recursive saturation of M, let $n > \mathbb{N}$ and $\sigma \in 2^n$ such that

$$\begin{aligned} Y(r_\sigma, s_\sigma) > d \wedge \forall i < n \; \Big(\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) \leqslant d\Big) \\ \wedge \forall i < n \; \Big([r_{\sigma\restriction_i}, s_{\sigma\restriction_i}] \supseteq [r_{\sigma\restriction_{i+1}}, s_{\sigma\restriction_{i+1}}]\Big). \end{aligned}$$

It can then be checked that $[\![r_\sigma, s_\sigma]\!] \subseteq [\![a, b]\!]$ is pregeneric over $\bar{c}$. □

One can try to strengthen the definition of pregeneric intervals to one involving tuples of length greater than two. However this does not give us anything much stronger, at least when the model is recursively saturated.

Proposition 5.10. *Suppose $\bar{c} \in M$ and M is recursively saturated. Then an interval $[\![a, b]\!] \in \mathcal{B}$ is pregeneric over $\bar{c}$ if and only if*

$$\forall \bar{x} \in [\![a, b]\!] \; \forall [\![u, v]\!] \subseteq [\![a, b]\!] \; \exists \bar{x}' \in [\![u, v]\!] \; (\bar{x}, \bar{c}) \equiv (\bar{x}', \bar{c}).$$

Proof. One direction is obvious. For the other, note that if we can deal with $\max\{\bar{x}\}$ and $\min\{\bar{x}\}$, then we can as well deal with the rest of $\bar{x}$ using recursive saturation. □

Remark. The above argument also shows that modulo recursive saturation, pregenericity of a $\mathcal{B}$-interval $[\![a, b]\!]$ over $\bar{c}$ in M is equivalent to

$$\forall [\![u, v]\!] \subseteq [\![a, b]\!] \; \exists a', b' \in [\![u, v]\!] \; (a, b, \bar{c}) \equiv (a', b', \bar{c}).$$

Another way to strengthen the notion of pregenericity is to require an interval to be pregeneric over all elements in a cut I. In some very particular cases, this works.

Proposition 5.11. *Suppose M is recursively saturated and $\mathcal{B} = \mathcal{B}^{\text{elem}}$. Let $c, c' \in M$, and let $[\![a,b]\!]$ be a finite elementary interval such that $c, c' \ll a$ and* $\operatorname{tp}(c) = \operatorname{tp}(c')$*. Then*

$$\forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists a', b' \in [\![u,v]\!]\ (a,b,c) \equiv (a',b',c').$$

In particular, if $c = c'$, then $[\![a,b]\!]$ is pregeneric over c.

Proof. Let $[\![a,b]\!]$ be a finite elementary interval, $c, c' \ll a$ and $[\![u,v]\!] \subseteq [\![a,b]\!]$.

First, we find $a' > u$ with $(a,c) \equiv (a',c')$ and $a' \ll v$. Consider the recursive type

$$\begin{aligned} p(x) = \{\varphi(x,c') \leftrightarrow \varphi(a,c) : \varphi(x,y) \in \mathscr{L}_{\mathrm{A}}\} \\ \cup \{t_n(x) < v : n \in \mathbb{N}\} \cup \{u < x\}. \end{aligned}$$

Take $n \in \mathbb{N}$ and $\varphi(x,y) \in \mathscr{L}_{\mathrm{A}}$ such that $M \vDash \varphi(a,c)$. Pick an elementary cut I in $[\![u,v]\!]$. Since $c \ll a$, we see that $M \vDash \mathsf{Q}x\ \varphi(x,c)$ where Q denotes 'there are cofinally many'. Our hypothesis on c and c' then implies that $M \vDash \mathsf{Q}x\ \varphi(x,c')$. By the elementarity of I in M, we have $M \vDash \mathsf{Q}x \in I\ \varphi(x,c')$. In particular, $M \vDash \exists x > u\ (t_n(x) < v \wedge \varphi(x,c'))$. So $p(x)$ is finitely satisfied in M. Using recursive saturation, let $a' \in M$ realise $p(x)$, so that

$$(a,c) \equiv (a',c') \text{ and } u < a' \ll v. \tag{$*$}$$

Next, consider the recursive type

$$q(y) = \{\theta(a,b,c) \leftrightarrow \theta(a',y,c') : \theta(x,y,z) \in \mathscr{L}_{\mathrm{A}}\} \cup \{y < v\}.$$

Let $\theta(x,y,z) \in \mathscr{L}_{\mathrm{A}}$ such that $M \vDash \theta(a,b,c)$. We need to show $M \vDash \exists y < v\ \theta(a',y,c')$. Now, we know that $M \vDash \exists y\, \theta(a,y,c)$ and so $M \vDash \exists y\, \theta(a',y,c')$ by $(*)$. Thus

$$(\mu y)(\theta(a',y,c')) \in \operatorname{cl}(a',c') \subseteq M_{\mathcal{B}}(\langle a',c'\rangle) < v,$$

proving that $q(y)$ is finitely satisfied in M. Using recursive saturation again, let b' realise $q(y)$ in M. Then

$$(a,b,c) \equiv (a',b',c') \text{ and } u < a' < b' < v,$$

as required. □

However, in most other cases, this does not work.

Proposition 5.12. *For every $B > \mathbb{N}$, there exists cofinally many $Y \in M$ such that, for every $\mathcal{B}$-interval $[\![a,b]\!] \subseteq M_{<B}$ and every $d > \mathbb{N}$, there exists a nonstandard $c < d$ with $[\![a,b]\!]$ not pregeneric over c, Y.*

Proof. Let $Y \in M$ be a monotone indicator for $\mathcal{B}$ below a sufficiently large $B \in M$. Let $[\![a,b]\!] \subseteq M_{<B}$ be a $\mathcal{B}$-interval and $d > \mathbb{N}$. Without loss of generality, suppose $Y(a,b) > d$. Using Lemma 2.15, pick $[\![u,v]\!] \subseteq [\![a,b]\!]$ such that $\mathbb{N} < Y(u,v) < d$. Let $c = Y(u,v)$. Then for all $[a',b'] \subseteq [\![u,v]\!]$, we have

$$Y(a',b') \leqslant Y(u,v) = c$$

by monotonicity of Y, and

$$Y(a,b) > d > Y(u,v) = c.$$

Hence $(a,b,c,Y) \not\equiv (a',b',c,Y)$ for every $[a',b'] \subseteq [\![u,v]\!]$. Therefore, $[\![a,b]\!]$ is not pregeneric over c, Y. □

These show that pregenericity is stable and optimal. More evidence of this comes from its relationship with arithmetic saturation.

Proposition 5.13. *If for every $f \in M$, there are $B \in M$ and an indicator Y for $\mathcal{B}$ below B such that a pregeneric interval over f, Y exists in $M_{<B}$, then $\mathbb{N}$ is strong in M.*

Proof. Suppose the hypothesis in the proposition holds. Let $f\colon \mathbb{N} \to M$ be a coded function in M. Abusing notation, we let f be a code for this function in M. Using the hypothesis, let $B \in M$ and Y be an indicator for $\mathcal{B}$ below B, and pick a $\mathcal{B}$-interval $[\![a,b]\!] \subseteq M_{<B}$ that is pregeneric over f, Y. Note that by the proof of Proposition 2.6, we may assume Y to be monotone.

We claim that $f(n) > \mathbb{N}$ if and only if $f(n) > Y(a,b)$ for all $n \in \mathbb{N}$. Note that since $[\![a,b]\!] \in \mathcal{B}$, the 'if part' is obvious. So let $n \in \mathbb{N}$ such that $f(n) > \mathbb{N}$. Using Lemma 2.15, let $[\![u,v]\!] \subseteq [\![a,b]\!]$ such that $\mathbb{N} < Y(u,v) < f(n)$. Recalling that $[\![a,b]\!]$ is pregeneric over f, Y, let $a', b' \in [\![u,v]\!]$ such that

$$(a,b,f,Y) \equiv (a',b',f,Y). \tag{†}$$

By monotonicity of Y, we have $Y(a',b') \leqslant Y(u,v) < f(n)$. Thus by (†), we get $Y(a,b) < f(n)$ as required. □

While pregeneric intervals are interesting in their own right, the reason for their introduction is to construct generic cuts. In doing this we shall prove the following characterisation of generic cuts in countable arithmetically saturated models.

Theorem 5.14. *Suppose M is countable and arithmetically saturated. A cut I is $\mathcal{Z}$-generic if and only if it is contained in a pregeneric $\mathcal{B}$-interval over $\bar{c}$ for every $\bar{c} \in M$.*

The proof of this will emerge in the discussion of this section. For the purpose of this proof, let us make the following temporary definition.

Definition 5.15. A cut is *generic′* if it is contained in a pregeneric $\mathcal{B}$-interval over $\bar{c}$ for every $\bar{c} \in M$.

It is easy to check that all generic′ cuts are in $\mathcal{Z}$. For if I is generic′ and $[a, b]$ is a semi-interval containing I, then there is an interval $[\![u, v]\!]$ pregeneric over a, b containing I. Now $a, b \notin [\![u, v]\!]$, and so $[a, b] \supseteq [\![u, v]\!]$. Hence $[a, b]$ is a $\mathcal{B}$-interval by axiom (4) for a neighbourhood system.

It is straightforward to show that generic′ cuts exist using Theorem 5.9 and Banach's characterisation of comeagre sets.

Theorem 5.16. *If M is countable and arithmetically saturated, then being a generic′ cut is an enforceable property of $\mathcal{Z}$-cuts.*

Proof. Let M be countable and arithmetically saturated. We play the Banach–Mazur game on $\mathcal{B}$. If $\bar{c} \in M$, then $\exists$ can make the outcome of a play be contained in a pregeneric interval over $\bar{c}$ using Theorem 5.9 in a single step. Since M is countable and $\exists$ has ω many steps to play, she can ensure that the outcome is contained in a pregeneric interval over $\bar{c}$ for every $\bar{c} \in M$. Therefore it is enforceable that the cut constructed is generic′. □

Corollary 5.17. *If M is countable and arithmetically saturated, then there exist generic′ cuts for $\mathcal{B}$. Furthermore every generic cut is generic′.*

A direct consequence of Proposition 5.13 and the definition of generic′ cuts is that the strength of $\mathbb{N}$ in the hypothesis of the above theorem is necessary.

Corollary 5.18. *If M contains a generic′ cut then $\mathbb{N}$ is strong in M.*

The other implication, that a generic′ cut is generic will follow from the next theorem, using Theorem 3.10.

Theorem 5.19. *Suppose M is countable and arithmetically saturated. Let $\bar{c} \in M$, and let $[\![a, b]\!] \in \mathcal{B}$ be a pregeneric interval over $\bar{c}$. Then any two generic′ cuts contained in $[\![a, b]\!]$ are conjugate over $\bar{c}$.*

Proof. We use a back-and-forth argument.

Let $\bar{c} \in M$, and let $[\![a, b]\!] \in \mathcal{B}$ be a pregeneric interval over $\bar{c}$. Pick two generic′ cuts I and I' in $[\![a, b]\!]$. At any stage of the back-and-forth, we have

- an interval $[\![u, v]\!]$ containing I,
- an interval $[\![u', v']\!]$ containing I', and
- tuples $\bar{r}, \bar{r}' \in M$

such that

- $[\![u,v]\!]$ is pregeneric over $\bar{c},\bar{r}$,
- $[\![u',v']\!]$ is pregeneric over $\bar{c},\bar{r}'$, and
- $(u,v,\bar{c},\bar{r}) \equiv (u',v',\bar{c},\bar{r}')$.

We show how to add an arbitrary ${}^*r$ to $\bar{r}$. In the process, we find ${}^*u, {}^*v$ to replace u,v and choose corresponding ${}^*u', {}^*v', {}^*r'$ while keeping $\bar{r}'$ fixed. This constitutes the 'forth' step. The 'back' step is similar.

Using the definition of generic$'$ cuts, choose an interval $[\![{}^*u, {}^*v]\!]$ that contains I and is pregeneric over $u,v,\bar{c},\bar{r},{}^*r$. Pick an automorphism $g \in \operatorname{Aut}(M,\bar{c})$ such that $\langle u,v,\bar{r}\rangle^g = \langle u',v',\bar{r}'\rangle$, which is possible since $(u,v,\bar{c},\bar{r}) \equiv (u',v',\bar{c},\bar{r}')$ and M is recursively saturated. It follows that $[\![{}^*u^g, {}^*v^g]\!] \subseteq [\![u',v']\!]$. Using pregenericity of $[\![u',v']\!]$ and recursive saturation, let $h \in \operatorname{Aut}(M,\bar{c},\bar{r}')$ such that $[\![u',v']\!]^h \subseteq [\![{}^*u^g, {}^*v^g]\!]$. The back-and-forth then continues by setting

$$[\![{}^*u', {}^*v']\!] = [\![{}^*u^{gh^{-1}}, {}^*v^{gh^{-1}}]\!] \text{ and } {}^*r' = {}^*rgh^{-1}.$$

The required isomorphism is given by $\bar{r} \mapsto \bar{r}'$ at the end. □

6 Conjugacy properties and truth

We continue working with a fixed neighbourhood system $\mathcal{B}$ and its associated species of cuts $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$. As in the previous section, we assume *$\mathcal{Z}$ to be order-isomorphic to 2^ω*. Additionally, in this section we assume that our model M *is countable and arithmetically saturated.*

Results in the last section show that, in this context, the set $\mathcal{G}$ of $\mathcal{Z}$-generic cuts is comeagre in $\mathcal{Z}$ and satisfies the hypotheses of Theorem 3.10. The neighbourhood of a generic cut is fuzzy or blurred in some sense, and this agrees with the idea that pregeneric intervals should be homogeneous. In fact, Theorem 3.10 says that this blurry nature actually characterises genericity. It is natural to ask exactly how large the blurry zone around a generic cut is. The following shows that one can improve Theorem 5.19 slightly.

Corollary 6.1. *If $\bar{c} \in M$ and $[\![a,b]\!]$ is an interval satisfying*

$$\exists x \in [\![a,b]\!]\ \forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x' \in [\![u,v]\!]\ (x,\bar{c}) \equiv (x',\bar{c}),$$

then all generic cuts in $[\![a,b]\!]$ are conjugate over $\bar{c}$.

Proof. Let $[\![a,b]\!]$ be an interval and $x,\bar{c} \in M$ such that

$$\forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x' \in [\![u,v]\!]\ (x,\bar{c}) \equiv (x',\bar{c}). \qquad (\ddagger)$$

Pick two generic cuts I_1 and I_2 from $[\![a,b]\!]$. Using Corollary 5.17, let $[\![u_1,v_1]\!]$ and $[\![u_2,v_2]\!]$ be pregeneric intervals over a,b,c that contain I_1 and I_2 respectively. Note that $[\![u_1,v_1]\!]$ and $[\![u_2,v_2]\!]$ have to be subintervals of $[\![a,b]\!]$.

Our plan is to map I_1 close enough to I_2 via x, so that Theorem 5.19 can be applied. Using the axioms for a neighbourhood system, let $[\![u_2',v_2']\!]$ be a pregeneric subinterval of $[\![u_2,v_2]\!]$ over c containing I_2 such that

$$u_2 \ll u_2' \ll v_2' \ll v_2. \tag{§}$$

Using (‡) and recursive saturation, let $g_1, g_2 \in \mathrm{Aut}(M,\bar{c})$ such that $x^{g_1} \in [\![u_1,v_1]\!]$ and $x^{g_2} \in [\![u_2',v_2']\!]$. It follows from (§) that $[\![u_1,v_1]\!]^{g_1^{-1}g_2} \cap [\![u_2,v_2]\!] \in \mathcal{B}$. By Theorem 5.19, both $I_1^{g_1^{-1}g_2}$ and I_2 are conjugate over $\bar{c}$ to the generic cuts in this intersection. Therefore, $(M,I_1,\bar{c}) \cong (M,I_2,\bar{c})$. □

This turns out to be the best possible.

Proposition 6.2. *Let $[\![a,b]\!]$ be a $\mathcal{B}$-interval, $\bar{c} \in M$, and $\mathcal{D} \subseteq \mathcal{Z}$ dense in $[\![a,b]\!]$. If all $\mathcal{D}$-cuts in $[\![a,b]\!]$ are conjugate over $\bar{c}$, then*

$$\exists x \in [\![a,b]\!]\ \forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x' \in [\![u,v]\!]\ (x,\bar{c}) \equiv (x',\bar{c}).$$

Proof. Using Theorem 5.9, let $[\![r,s]\!] \subseteq [\![a,b]\!]$ be a pregeneric interval of $\bar{c}$, and pick $x \in [\![r,s]\!]$. We show that this x works.

Let $[\![u,v]\!] \subseteq [\![a,b]\!]$ be arbitrary. We apply a similar trick as in the previous proof again. Using the axioms for a neighbourhood system, let $[\![u',v']\!]$ be a subinterval of $[\![u,v]\!]$ such that

$$u \ll u' \ll v' \ll v.$$

Using the density of $\mathcal{D}$ in $[\![a,b]\!]$, take $\mathcal{D}$-cuts $I \in [\![r,s]\!]$ and $J \in [\![u',v']\!]$. By assumption, I is conjugate to J over $\bar{c}$. Let $h \in \mathrm{Aut}(M,\bar{c})$ such that $I^h = J$. Then $[\![r,s]\!]^h \cap [\![u,v]\!]$ is an interval whose preimage under h is a subinterval of $[\![r,s]\!]$. Let $[\![r',s']\!]$ be this preimage. Recall that $[\![r,s]\!]$ is a pregeneric interval over $\bar{c}$. So there exists an automorphism $g \in \mathrm{Aut}(M,\bar{c})$ such that $x^g \in [\![r',s']\!]$ and hence $x^{gh} \in [\![u,v]\!]$, as required. □

We now start to prove some new results that have no counterparts in GCMA. The main theorem is a syntactic characterisation of conjugacy for generic cuts. As a corollary, we obtain a description of the orbits of M under the action of $\mathrm{Aut}(M,I)$ where I is a generic cut.

Our first objective is to count the number of conjugacy classes of generic cuts. It will turn out that in some cases there will be exactly $\aleph_0$ conjugacy classes, and in other cases just one. We have already proved results showing that under certain conditions two generic cuts are conjugate. To characterise conjugacy, we additionally need to know when two generic cuts are not conjugate. It is obvious that if two cuts are separated by a definable point, then they cannot be conjugate, and this observation gives us one set of examples.

Example 6.3. Let $\mathcal{D}$ be a dense set in $\mathcal{Z}$ that is invariant under the action of $\mathrm{Aut}(M)$, and suppose $\mathcal{B} = \mathcal{B}^Y$ for some GCMA indicator Y. If $M \not\models \mathrm{Th}(\mathbb{N})$, then there are at least countably infinitely many conjugacy classes of $\mathcal{D}$-cuts that are contained in $\overline{\mathrm{cl}}(0)$.

Proof. Let $\mathcal{D}$, Y and $\mathcal{B} = \mathcal{B}^Y$ be as in the statement and suppose $M \not\models \mathrm{Th}(\mathbb{N})$. Since $\mathcal{Z}$ is closed, $M_{\mathcal{B}}(0)$ exists and is in $\mathcal{Z}$. Note that

$$M_{\mathcal{B}}(0) = \sup\{(\mu y)(Y(0, y) \geqslant n) : n \in \mathbb{N}\} \subsetneq_{\mathrm{e}} \overline{\mathrm{cl}}(0).$$

Take $a \in \mathrm{cl}(0)$ such that $a > M_{\mathcal{B}}(0)$. Then $[\![0, a]\!] \in \mathcal{B}$ by the definition of $M_{\mathcal{B}}(0)$. Using an argument similar to that in the proof of Proposition 2.3 one can divide the $\mathcal{B}$-interval $[\![0, a]\!]$ indefinitely into smaller subintervals by definable points. Since $\mathcal{D}$ is dense in $\mathcal{Z}$, we get any finite number of mutually non-conjugate $\mathcal{D}$-cuts in $\overline{\mathrm{cl}}(0)$. □

When $M \models \mathrm{Th}(\mathbb{N})$, this trick does not work because there is no nonstandard definable point. Instead we may make use of a function H that grows like an ascending sequence of gaps. The cuts in consecutive gaps cannot be conjugate because the maximum w such that $H(w)$ is in the cut are all in different congruence classes modulo a sufficiently large natural number. The following technical lemma allows this to work.

Lemma 6.4. *Let Y be a GCMA indicator. If $M \models \forall x \exists y\, Y(x, y) \geqslant n$ for each $n \in \mathbb{N}$, then there is a strictly increasing function $H \colon M \to M$ definable in M without parameters such that*

$$H(k) \ll_{\mathcal{B}^Y} H(k+1)$$

for all large enough $k \in M$.

Proof. Suppose Y is as in the hypothesis.

If $M \models \forall n \forall x \exists y\, Y(x, y) \geqslant n$, then let H be the function defined recursively by

$$H(0) = 0 \wedge \forall z \Big(H(z+1) = (\mu y)(Y(H(z), y) \geqslant z+1)\Big).$$

If $M \models \exists n \exists x \forall y\, Y(x, y) < n$, then define H by

$$H(0) = 0 \wedge \forall z \Big(H(z+1) = (\mu y)(Y(H(z), y) \geqslant n)\Big),$$

where $n = (\max m)(\forall x \exists y\, Y(x, y) \geqslant m)$. □

Proposition 6.5. *Let Y be a GCMA indicator such that $\mathcal{B} = \mathcal{B}^Y$, and let $\mathcal{D}$ be a dense set of $\mathcal{Z}$-cuts that is closed under the action of* $\mathrm{Aut}(M)$*.*

(a) If $M \nvDash \forall x \exists y\, Y(x, y) \geqslant n$ for some $n \in \mathbb{N}$, then no $\mathcal{Z}$-cut can contain $\overline{\mathrm{cl}}(0)$.
(b) If $M \vDash \forall x \exists y\, Y(x, y) \geqslant n$ for all $n \in \mathbb{N}$, then there are at least countably infinitely many mutually non-conjugate $\mathcal{D}$-cuts containing $\overline{\mathrm{cl}}(0)$.

Proof. Let Y and $\mathcal{D}$ be as in the statement of the proposition.

For (a), take $n \in \mathbb{N}$ such that $M \vDash \exists x \forall y\, Y(x, y) < n$. Let x^* be the least x such that $M \vDash \forall y\, Y(x, y) < n$. Then $x^* \in \mathrm{cl}(0)$ and no $\mathcal{B}$-interval is above x^* because $n \in \mathbb{N}$. So, there cannot be any $\mathcal{Z}$-cut above $\overline{\mathrm{cl}}(0)$.

For (b), suppose $M \vDash \forall x \exists y\, Y(x, y) \geqslant n$ for each $n \in \mathbb{N}$. Let H be a fast growing function whose existence is guaranteed by Lemma 6.4. Pick $x > \overline{\mathrm{cl}}(0)$ such that $\big(\llbracket H(x + k), H(x + k + 1)\rrbracket\big)_{k\in\mathbb{N}}$ is a sequence of $\mathcal{B}$-intervals, which is possible by recursive saturation. Using the density of $\mathcal{D}$ in $\mathcal{Z}$, take a $\mathcal{D}$-cut $I_k \in \llbracket H(x + k), H(x + k + 1)\rrbracket$ for each $k \in \mathbb{N}$. Noting that

$$(\max w)(H(w) \in I_k) = x + k$$

for each $k \in \mathbb{N}$, it can easily be verified that the cuts in $(I_k)_{k\in\mathbb{N}}$ are mutually non-conjugate. □

Corollary 6.6. *If $\mathcal{B} = \mathcal{B}^Y$ for some GCMA indicator Y, then there are exactly countably infinitely many conjugacy classes of generic cuts in M.*

Proof. Recall that Theorem 5.19 says that if two generic cuts are in the same pregeneric interval, then they are conjugate. By the countability of M, this implies that there can be at most countably infinitely many conjugacy classes of generic cuts in M.

On the other hand, note that it is not possible to have $M \vDash \mathrm{Th}(\mathbb{N})$ and

$$M \vDash \exists x \forall y\, Y(x, y) < n \text{ for some } n \in \mathbb{N}$$

both true at the same time. Otherwise, the truth of $\exists x \exists y\, Y(x, y) \geqslant n$ in M for every $n \in \mathbb{N}$ then implies the existence of a nonstandard definable element. Therefore we are done by Example 6.3, Proposition 6.5, Theorem 5.16, and the Baire Category Theorem. □

Remark. Note that there is exactly one conjugacy class of generic cuts for $\mathcal{B}^{\mathrm{elem}}$ by Theorem 5.19 and Proposition 5.11.

All the above non-conjugacy claims are actually proved by cooking up a sentence that is true in one structure but not the other. One may ask whether we are able to find non-conjugate generic cuts that are elementary equivalent in the expanded language. The following suggests that this may not be possible.

Example 6.7. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$, and let I be a generic cut. If $a, b \in I$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$, then $(M, I, a) \cong (M, I, b)$.

Proof. Let $a, b \in I \prec_{\mathrm{e}} M$ such that I is generic and $\mathrm{tp}(a) = \mathrm{tp}(b)$. Using Theorem 5.14, let $[\![r, s]\!]$ be a pregeneric interval over a, b that contains I. Then we necessarily have $a, b \ll r$.

Using Proposition 5.11 and recursive saturation, let $g \in \mathrm{Aut}(M)$ such that

$$a = b^g \text{ and } [\![r, s]\!]^g \subseteq [\![r, s]\!].$$

Let $J = I^g$. Then

$$J = I^g \in [\![r, s]\!]^g \subseteq [\![r, s]\!]$$

so that both I and J are generic cuts in $[\![r, s]\!]$. However, $[\![r, s]\!]$ is pregeneric over a by Proposition 5.11. So by Theorem 5.19, there is an automorphism $h \in \mathrm{Aut}(M, a)$ such that $J^h = I$ and thus

$$(M, I, b) \cong (M, I^g, b^g) = (M, J, a) \cong (M, J^h, a^h) = (M, I, a),$$

as required. □

This essentially says that the $\mathscr{L}_{\mathrm{A}}^I$ formula '$x \in I$' tells us a lot about an element x when I is generic for $\mathcal{B}^{\mathrm{elem}}$. On the other hand, the formula '$x \notin I$' is much weaker.

Proposition 6.8. *Suppose that all $\mathcal{Z}$-cuts are closed under addition and multiplication. If I is a generic cut, $c \in M$ and $B > I$, then there are $d, d' \in M$ such that $I < d, d' < B$ and $(d, c) \equiv (d', c)$, but $(M, I, d, c) \not\equiv (M, I, d', c)$.*

Proof. Under the hypotheses of the proposition, let $[\![a, b]\!] \in \mathcal{B}$ be a pregeneric interval over c, B containing I using Theorem 5.14.

By Proposition 4.2 and Theorem 5.14, $I \neq M_{\mathcal{B}}[b]$, so $I < M_{\mathcal{B}}[b]$. Let $w \in M_{\mathcal{B}}[b] \setminus I$. By Proposition 2.3, $M_{\mathcal{B}}(w) \neq M_{\mathcal{B}}[b]$. Take $z \in M_{\mathcal{B}}[b] \setminus M_{\mathcal{B}}(w)$ and let $d = \langle w, z \rangle$. Note that $M_{\mathcal{B}}[b] \in \mathcal{Z}$ is closed under addition and multiplication, and thus $d \in M_{\mathcal{B}}[b]$. So now, we have

$$a \in I < w \ll z < \langle w, z \rangle = d \in M_{\mathcal{B}}[b] < b.$$

Using Theorem 5.16 and the Baire Category Theorem, pick a generic cut $J \in [\![w, z]\!] \subseteq [\![a, b]\!]$. Then I and J are conjugate over c, B by Theorem 5.19. Let $g \in \mathrm{Aut}(M, c, B)$ such that $J^g = I$. Let $d' = d^g$ so that $(d, c, B) \equiv (d', c, B)$. In particular, as $d < B$, we have $d' < B$ as well. Note also that since $J < d$, we have

$$I = J^g < d^g = d'.$$

Let π_L be the Skolem function defined by

$$\forall p\ \Big(\pi_L(p) = (\mu x)\Big(\exists y(p = \langle x, y\rangle)\Big)\Big).$$

Then $\pi_L(d) = \pi_L(\langle w, z\rangle) = w > I$, but since $w \in J$, we have

$$\pi_L(d') = \pi_L(d^g) = (\pi_L(d))^g = w^g \in J^g = I.$$

Therefore, $(M, I, d, c) \not\equiv (M, I, d', c)$. □

Again, the above proof uses an $\mathscr{L}_{\mathrm{A}}^I$ formula that is true in one structure but not in the other to prove non-conjugacy. This seems to provide evidence supporting the conjecture that the $\mathscr{L}_{\mathrm{A}}^I$ theory of (M, I) determines its conjugacy class when I is generic. We shall now show that this conjecture is in fact true. Surprisingly, the formulas used in the proof of Proposition 6.5 are already sufficient to describe the theory of (M, I). The next definition sets up the notation we shall need properly.

Definition 6.9. Let $\varphi(\bar{x}, y)$ be an $\mathscr{L}_{\mathrm{A}}$ formula, $I \in \mathcal{C}$ and $\bar{c} \in M$. We write $\nu^I_{\varphi(\bar{x},y)}(\bar{c})\!\downarrow$ for

$$\exists y \in I\ \Big(\varphi(\bar{c}, y) \wedge \forall y' \in I\ (y' > y \to \neg\varphi(\bar{c}, y'))\Big).$$

The expression $\nu^I_{\varphi(\bar{x},y)}(\bar{c})\!\uparrow$ is the negation of $\nu^I_{\varphi(\bar{x},y)}(\bar{c})\!\downarrow$. Define

$$\nu^I_{\varphi(\bar{x},y)}(\bar{c}) = \begin{cases} (\max y \in I)(\varphi(\bar{c}, y)), & \text{if } \nu^I_{\varphi(\bar{x},y)}(\bar{c})\!\downarrow; \\ 0, & \text{otherwise.} \end{cases}$$

Note that the statements defined above can be expressed in the language $\mathscr{L}_{\mathrm{A}}^I$.

Lemma 6.10. *Let $I \in \mathcal{Z}$ be generic. If $\bar{c} \in M$, and $[\![a, b]\!] \in \mathcal{B}$ is pregeneric over $\bar{c}$ and contains I, then $\nu^I_{\varphi(\bar{x},y)}(\bar{c}) < a$ for every $\mathscr{L}_{\mathrm{A}}$ formula $\varphi(\bar{x}, y)$ such that $\nu^I_{\varphi(\bar{x},y)}(\bar{c})\!\downarrow$.*

Proof. Fix an $\mathscr{L}_{\mathrm{A}}$ formula $\varphi(\bar{x}, y)$. Clearly $0 < a$. Suppose $M \models \nu^I_{\varphi(\bar{x},y)}(\bar{c})\!\downarrow$. Let $A = \nu^I_{\varphi(\bar{x},y)}(\bar{c}) + 1 \in I$. Then $M_{\mathcal{B}}(A) < I$ by Proposition 4.2 and Theorem 5.14. Let $B \in M$ such that $M_{\mathcal{B}}(A) < B \in I$. If $A > a$, then $[\![A, B]\!] \subseteq [\![a, b]\!]$ and

$$M \models \nu^I_{\varphi(\bar{x},y)}(\bar{c}) \in [\![a, b]\!] \wedge \varphi(\bar{c}, \nu^I_{\varphi(\bar{x},y)}(\bar{c}))$$

while $M \models \forall y \in [\![A, B]\!]\ \neg\varphi(\bar{c}, y)$ by the maximality of $\nu^I_{\varphi(\bar{x},y)}(\bar{c})$, which is not possible since $[\![a, b]\!]$ is pregeneric over $\bar{c}$. Therefore, $\nu^I_{\varphi(\bar{x},y)}(\bar{c}) < A \leqslant a$. □

Theorem 6.11. *Suppose M is countable and arithmetically saturated. Let $\bar{c} \in M$, and let $I, J \in \mathscr{Z}$ be generic. Then $(M, I, \bar{c}) \cong (M, J, \bar{c})$ if and only if for every $\mathscr{L}_\mathrm{A}$ formula $\alpha(\bar{x}, y)$*

$$(M, I) \vDash \nu^I_{\alpha(\bar{x},y)}(\bar{c})\downarrow \Leftrightarrow (M, J) \vDash \nu^J_{\alpha(\bar{x},y)}(\bar{c})\downarrow.$$

Proof. One direction is obvious. For the other direction, let $\bar{c} \in M$, and let $I, J \in \mathscr{Z}$ be generic such that $M \vDash \nu^I_{\alpha(\bar{x},y)}(\bar{c})\downarrow \leftrightarrow \nu^J_{\alpha(\bar{x},y)}(\bar{c})\downarrow$ for every $\mathscr{L}_\mathrm{A}$ formula $\alpha(\bar{x}, y)$. Without loss of generality, assume $I < J$. Using Corollary 5.17, pick a pregeneric interval $[\![a, b]\!]$ over $\bar{c}$ containing I, and a pregeneric interval $[\![u, v]\!]$ over $\bar{c}$ containing J. By genericity and Proposition 4.2 we have $M_\mathcal{B}(a) < I$. Let $A \in M$ such that $M_\mathcal{B}(a) < A \in I < b$.

Consider the recursive type

$$p(y) = \{u \leqslant y \leqslant v\} \cup \{\alpha(\bar{c}, y) \leftrightarrow \alpha(\bar{c}, A) : \alpha(\bar{x}, y) \in \mathscr{L}_\mathrm{A}\}.$$

We show that this is finitely satisfied in M. Let $\alpha(\bar{x}, y) \in \mathscr{L}_\mathrm{A}$ such that $M \vDash \alpha(\bar{c}, A)$. Now if $M \vDash \nu^I_{\alpha(\bar{x},y)}(\bar{c})\downarrow$, then by Lemma 6.10 and the maximality of $\nu^I_{\alpha(\bar{x},y)}(\bar{c})$, we have

$$a \ll A \leqslant \nu^I_{\alpha(\bar{x},y)}(\bar{c}) < a,$$

which is a contradiction. So $M \vDash \nu^I_{\alpha(\bar{x},y)}(\bar{c})\uparrow$. By our hypothesis, we have $M \vDash \nu^J_{\alpha(\bar{x},y)}(\bar{c})\uparrow$. Note that $A \in I < J$ and $M \vDash \alpha(\bar{c}, A)$. So there are cofinally many $y \in J$ such that $M \vDash \alpha(\bar{c}, y)$. In particular, there is a $y \in J$ such that $M \vDash y \geqslant u \wedge \alpha(\bar{c}, y)$. Thus $M \vDash \exists y \in [\![u, v]\!] \ \ \alpha(\bar{c}, y)$, as required.

Let B realise $p(y)$ in M. By construction, $\mathrm{tp}(A, \bar{c}) = \mathrm{tp}(B, \bar{c})$. Using recursive saturation of M, let $g \in \mathrm{Aut}(M, \bar{c})$ such that $A^g = B \in [\![u, v]\!]$. Since $a \ll A \ll b$, the intersection $[\![a, b]\!]^g \cap [\![u, v]\!]$ is a $\mathcal{B}$-interval. Using Theorem 5.16 and the Baire Category Theorem, pick a generic cut J' in this interval. By Theorem 5.19, J is conjugate to J' over $\bar{c}$, and $(J')^{g^{-1}}$ is conjugate to I over $\bar{c}$. Therefore, I is conjugate to J over $\bar{c}$. $\square$

Apart from giving alternative proofs of Proposition 6.2 and Example 6.7 for generic cuts, this theorem also implies a weak quantifier elimination result.

Definition 6.12. Define $\mathscr{L}^I_\nu$ to be the language obtained from $\mathscr{L}^I_\mathrm{Sk}$ by adding a new predicate

$$\nu^I_{\alpha(\bar{x},y)}(\bar{x})\downarrow$$

for each $\mathscr{L}_\mathrm{A}$ formula $\alpha(\bar{x}, y)$. $\mathscr{L}^I_\mathrm{A}$ structures are interpreted as $\mathscr{L}^I_\nu$ structures in the natural way.

Corollary 6.13. *Let $I \in \mathscr{Z}$ be generic and $\bar{a}, \bar{b} \in M$. Then $(M, I, \bar{a}) \cong (M, I, \bar{b})$ if and only if $\bar{a}$ and $\bar{b}$ satisfy the same quantifier free $\mathscr{L}^I_\nu$ formulas with respect to I. In particular, (M, I) is ω-homogeneous.*

The following example shows that the new predicates $\nu^I_{\alpha(\bar{x},y)}(\bar{x})\!\downarrow$ are necessary for the previous corollary. The idea is very similar to that in Proposition 6.5(b).

Example 6.14. Suppose $\mathcal{B} = \mathcal{B}^{\text{elem}}$, and let $I \in \mathscr{Z}$ be generic. Then the formula

$$(\max j)((x)_j \in I) \text{ is even},$$

which is equivalent to

$$\exists w\Big((x)_{2w} = \nu^I_{\exists j(y=(x)_j)}(x)\Big),$$

is not equivalent in (M, I) to a quantifier-free $\mathscr{L}^I_{\text{Sk}}$ formula. In fact, it is not even equivalent to any infinite conjunction of quantifier-free $\mathscr{L}^I_{\text{Sk}}$ formulas.

Proof. Using recursive saturation, let $c \in M$ code an ascending sequence of gaps of length ω, i.e., c codes a sequence of nonstandard length such that $(c)_i \ll (c)_{i+1}$ for each $i \in \mathbb{N}$. Let $l \in M$ be the length of this sequence. Without loss of generality, assume this sequence is strictly increasing on its domain. Pick an indicator Y for $\mathcal{B}$ below $\max_{i<l}(c)_i + 1$. Using the strength of $\mathbb{N}$ in M, let $\nu \in M$ be nonstandard such that

$$Y((c)_i, (c)_{i+1}) > \mathbb{N} \Leftrightarrow Y((c)_i, (c)_{i+1}) > \nu.$$

for every $i \in \mathbb{N}$. By overspill, let $m > \mathbb{N}$ such that

$$\forall i < m \;\; Y((c)_i, (c)_{i+1}) > \nu.$$

Using arithmetic saturation, let $i < m$ be nonstandard such that $i \notin \text{cl}(c)$.

Pick generic cuts $I \in [\![(c)_{i-1}, (c)_i]\!]$ and $J \in [\![(c)_i, (c)_{i+1}]\!]$. Notice that Proposition 5.11 and Theorem 5.19 imply that I and J are conjugate. Let $g \in \text{Aut}(M)$ such that $I = J^g$ and set $d = c^g$. Then by our choices of I and J,

$$(\max j)((c)_j \in I) \text{ and } (\max j)((c)_j \in J)$$

are of different parities. Hence

$$(M, I, c) \not\equiv (M, J, c) \cong (M, J^g, c^g) = (M, I, d).$$

On the other hand, if t is a Skolem function such that $t(c) \in [\![(c)_{i-1}, (c)_{i+1}]\!]$, then i is definable from $(\mu j)((c)_j \geqslant t(c)) \in \text{cl}(c)$, which is contradictory to our choice of i. So for every Skolem function t, we either have $t(c) < (c)_{i-1}$, or $(c)_{i+1} < t(c)$. It follows that

$$t(c) \in I \text{ iff } t(c) < (c)_{i-1} \text{ iff } t(c) \in J \text{ iff } t(c^g) \in J^g \text{ iff } t(d) \in I$$

for every Skolem function t in $\mathscr{L}_{\mathrm{A}}$. Thus, c and d have the same quantifier-free $\mathscr{L}_{\mathrm{Sk}}^{I}$ type since $c^g = d$. Therefore, it is not possible that the formula

$$(\max j)((x)_j \in I) \text{ is even}$$

is equivalent to an infinite conjunction of quantifier-free $\mathscr{L}_{\mathrm{Sk}}^{I}$ formulas. $\square$

We are not yet able to prove a real quantifier elimination result, and whether such a result is possible is the main open question arising from this work.

Question 6.15. Let $\mathcal{Z}$ be a closed species of cuts without isolated points and I be a $\mathcal{Z}$-generic cut. Is it the case that every $\mathscr{L}_{\mathrm{A}}^{I}$ formula $\theta(\bar{x})$ is equivalent in (M, I) to a single quantifier-free formula $\theta_{\mathrm{qf}}(\bar{x})$ in the language $\mathscr{L}_{\nu}^{I}$ with the same free variables?

The main obstruction to answering this question at present is the observation that (M, I) is not recursively saturated even for types built from rather simple $\mathscr{L}_{\nu}^{I}$ formulas.

7 Elementary generic cuts

Elementary cuts are so important and often studied that we feel it useful to highlight them as a special case of the general theory above. Throughout this section we assume that our model M *of* PA *is countable and arithmetically saturated.*

In the case when $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$ and $\mathcal{Z} = \mathcal{Z}^{\mathrm{elem}} = \mathcal{Z}(\mathcal{B})$ of Example 2.13, we have shown that generic cuts for this species exist; we shall call these cuts *elementary generic cuts*. The notation $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ will be used without the subscripts following the convention in the literature.

One useful property of the neighbourhood system of elementary intervals is the following.

Proposition 7.1. *The notion of elementary intervals $\mathcal{B}^{\mathrm{elem}}$ is relatively indestructible.*

Proof. Let $[\![a, b]\!] \in \mathcal{B}$. Consider the recursive type

$$p(x) = \{\forall i < a\ (t_n((x)_i) < (x)_{i+1}) : n \in \mathbb{N}\} \cup \{(x)_0 = a \wedge (x)_a \leqslant b\}.$$

This is finitely satisfied in M since $[\![a, b]\!]$ contains an elementary cut. Any element realizing $p(x)$ in M witnesses the relative indestructibility of $[\![a, b]\!]$. $\square$

Therefore, by Propositions 4.8 and 4.9, an elementary generic cut is semiregular but not regular in M. It follows that M is never a conservative extension of an elementary generic cut I, since I would be strong and hence regular in any conservative extension.

Elementary generic cuts, like generic cuts for other species, are not definable over a finite set of parameters in any logic. This gives an alternative way to prove, for instance, Proposition 4.2, which says that elementary generic cuts cannot be of the form $M(a)$ or $M[b]$. Using Corollary 4.6 and the well-known idea of chronic resplendency (see for example the presentation in Kaye [1, Theorem 15.8]) it is also easy to see that there is no Σ_1^1 formula characterising genericity below any $B \in M$.

Proposition 4.2 gives us some information about automorphisms fixing I pointwise via a theorem by Kotlarski [11].

Theorem 7.2 (Kotlarski [11, Theorem 4.1])**.** *Let J be an elementary cut of M. If $J \neq M[b]$ for any $b \in M$, then J is* closed *in M, i.e.,*

$$\forall b > J \ \ \exists g \in \mathrm{Aut}(M) \ \Big(\forall x \in J \ x^g = x \ \textit{and} \ b^g \neq b\Big).$$

Corollary 7.3. *All elementary generic cuts are closed.*

It also follows from Proposition 4.2 and and Lemmas 2 and 4 of Kotlarski [10] that an elementary generic cut I of M is recursively saturated as an $\mathscr{L}_{\mathrm{A}}$ structure. The standard systems of I and M are the same, since I is nonstandard, and so, by general results, I and M are isomorphic. This proves the following.

Proposition 7.4. *There is a countable arithmetically saturated elementary end-extension N of M such that M is elementary generic in N.*

Similarly, any countable arithmetically saturated M is $K[b]$ for some countable arithmetically saturated elementary end-extension K of M and some $b \in K$. So we have the following.

Proposition 7.5. *There is an elementary end-extension N of M such that M is not elementary generic in N.*

Although an elementary generic cut I is 'rich' considered as a model in its own right, the pair of models (M, I) is not recursively saturated as an $\mathscr{L}_{\mathrm{A}}^I$ structure (Corollary 4.6). The proof of that corollary gives an example of a recursive set of formulas that is finitely satisfied but not realised. It is instructive in the case of elementary generic cuts to give an alternative example.

The idea of *sequences of skies* or *gaps*, introduced by Smoryński–Stavi [16] and discussed further by Smoryński [14] and Kossak–Schmerl [9], gives us a particularly nice necessary condition on (M, J) being recursively saturated, where J is an elementary cut of M.

Fact 7.6 (Smoryński [14, Theorem 2.8])**.** *If J is an elementary cut such that (M, J) is recursively saturated as an $\mathscr{L}_{\mathrm{A}}^{I}$ structure, then J is the limit of an ascending sequence of gaps of length J.*

Proposition 7.7. *An elementary generic cut I of M is not the limit of an ascending sequence of gaps of length I.*

Proof. Suppose $c \in M$ codes an ascending sequence of gaps of length I such that

$$\sup\{(c)_i : i \in I\} = I.$$

Using Corollary 5.17, pick a pregeneric interval $[\![a, b]\!] \in \mathcal{B}$ over c that contains I. Note that the sequence $((c)_i)_{i \in I}$ is cofinal in I. So let $i \in I$ such that $(c)_i > a$. By Theorems 5.16 and 5.19, I is conjugate to a generic cut in $[\![(c)_i, (c)_{i+1}]\!] \subseteq [\![a, b]\!]$ over c. This is impossible since no $\mathscr{Z}$-cut $J \in [\![(c)_i, (c)_{i+1}]\!]$ can satisfy

$$\{(c)_j \in J : j \in M \text{ is less than the length of } c\} \subseteq_{\mathrm{cf}} J,$$

as required. □

All our known examples of elements $c \in M$ for which $\mathbb{N}$ is definable in (M, I, c) are above I. So we ask the following.

Question 7.8. Suppose I is elementary generic. What is the set

$$\{c \in M : \mathbb{N} \text{ is definable in } (M, I, c)\}?$$

In particular, is it a subset of $M \setminus I$?

We conjecture that the elements of M definable in (M, I, c) are precisely the elements in the Skolem closure of $\{\nu^{I}_{\alpha(x,y)}(c) : \alpha(x, y) \in \mathscr{L}_{\mathrm{A}}\}$. In the case when c is absent, by using Corollary 7.12 below and a theorem by Kossak and Bamber [8], one can verify that all elements definable without parameters in (M, I) are in $\mathrm{cl}(0)$.

Theorem 7.9 (Kossak and Bamber [8, Theorem 4.1])**.** *If $J \in \mathcal{C}$ is closed under exponentiation, then every element definable in (M, J) without parameters is in* $\mathrm{cl}(c)$ *for some $c \in J$.*

To return to the topic of conjugacy properties, let us recall the following particular case of Proposition 5.11.

Proposition 7.10. *Every elementary interval has a pregeneric subinterval that contains exactly the same elementary cuts.*

A consequence of this is Example 6.7, which says that

$$\forall a, b \in I \left(\mathrm{tp}(a) = \mathrm{tp}(b) \Rightarrow (M, I, a) \cong (M, I, b) \right)$$

for an elementary generic cut I. This relates generic cuts to the notion of *free cuts* defined by Kossak.

Definition 7.11 (Kossak [6,7])**.** An elementary cut I is *free* if whenever $a, b \in I$ with $\mathrm{tp}(a) = \mathrm{tp}(b)$, we have $(M, I, a) \equiv (M, I, b)$.

Corollary 7.12. *All elementary generic cuts are free.*

This provides new examples of free cuts. By Theorem 5.19 and Proposition 5.11, all elementary generic cuts are conjugate, and hence by Theorem 5.16 the orbit of I under the action of $\mathrm{Aut}(M)$ has cardinality $2^{\aleph_0}$. This partially answers a question by Kossak [7, Problem 4.7]. Proposition 6.8 also says something about the degree of freeness of I. In Kossak's terminology [6], it says that I is the largest initial segment J of M such that I is J*-free* in M.

However, in view of the above discussion, this does not provide us with an example of a free cut I such that (M, I) is recursively saturated. One possible way to pursue this problem is to relax the axioms for a neighbourhood system so that Proposition 7.7 cannot be proved but enough freeness is retained. The statement of Proposition 2.3 seems to be a good candidate for a weakening of axiom (5). Another way is to use arguments similar to those in Section 6 of GCMA. A positive answer to the following question will also help.

Question 7.13. If M is arithmetically saturated, I is generic for some species, and $\bar{a} \in M$, is the theory $\mathrm{Th}(M, I, \bar{a})$ coded in M?

In view of the interesting work that has been done on the automorphism group of a countable recursively saturated or arithmetically saturated model of PA, it would seem that the automorphism group $\mathrm{Aut}(M, I)$ is begging to be explored, where I is elementary generic, or more generally generic in some other closed species of cuts. Theorem 5.19 and Corollary 6.13 provide useful ways to construct automorphisms in this group. The new back-and-forth system taken from GCMA, together with the well-known ones, suggest that the structure of such groups is quite rich.

We only state two questions relating to this group here, and leave it to the reader's imagination to come up with others. In the next two questions, let I be elementary generic in M, or perhaps generic in some other closed species $\mathcal{Z}$.

Question 7.14. Is $\mathrm{Aut}(M, I)$ a maximal subgroup of $\mathrm{Aut}(M)$?

Note that $\mathrm{Aut}(M, I)$ is naturally equipped with a topology, namely that generated by cosets of pointwise stabilisers of finite tuples from M. It is straightforward to see that $G_{(I)}$ is a closed normal subgroup of $G_{\{I\}}$.

Question 7.15. Other than $G_{(I)}$, what are the other closed normal subgroups of $G_{\{I\}}$? In particular, if $M \models \mathrm{Th}(\mathbb{N})$, is $G_{(I)}$ the only closed normal subgroup of $G_{\{I\}}$?

Another topic that is worth looking into is about $\mathscr{L}_{\mathrm{A}}^{I}$ elementary extensions of the structure (M, I), where I is elementary generic in M. By standard model theoretic techniques, we know that there is a countable elementary extension of (M, I) that is recursively saturated in the expanded language. So genericity is not preserved in all such extensions by Corollary 4.6. However, is there any proper elementary extension $(N, J) \succ (M, I)$ such that J is generic in N?

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