GENERIC CUTS IN A GENERAL SETTING

by

TIN LOK WONG

A thesis submitted to

The University of Birmingham

for the degree of

MASTER OF PHILOSOPHY (QUALIFYING)

School of Mathematics

The University of Birmingham

May 2008

Abstract

Recently Kaye (2008) axiomatized the notion of indicators and discovered that for each indicator there is a nice family of cuts in models of arithmetic called the generic cuts. This thesis extends his work in two ways. Firstly, we generalize the idea of indicator to a related notion of intervals that extends Kaye's theory to one that includes the case of elementary cuts. Then most results in Kaye's paper transfer to the present context. In particular we obtain a notion of elementary generic cuts, which are studied in more detail. These provide a partial answer to a question by Kossak (1995). Secondly, some new results are presented. The main one is a quantifier elimination theorem involving the structure (M, I) where I is a generic cut (relative to a notion of intervals) in a model M of Peano arithmetic.

ACKNOWLEDGEMENTS

Most important of all, I want to thank my supervisor and teacher, Dr. Richard Kaye, for his continuous support, patience, and the logic he taught me. He guided me through my journey in models of arithmetic, and showed me a way of doing mathematical research. All the work presented in this thesis is joint work with my supervisor.

I also want to express my gratitude towards every one I met in the School of Mathematics (and Statistics), past or present, in the University of Birmingham. Amongst other things, they taught me what mathematics really is, how one distinguishes between beautiful and ugly mathematics, and in what attitude should one view mathematics.

My wife Celia and my family are an important and integral part of my life. Their all round support has always been a great source of motivation and backup for me. They shaped every good aspect of my personality, and gave me many fond memories. Without them, I would never have been in a position to write this thesis. I want to thank them for all these.

Finally, I wish to thank the Overseas Research Students Awards Scheme (ORSAS) and the School of Mathematics for their financial support during my studies.

This thesis is typeset using LaTeX with the help of AMS-LaTeX, mathrsfs, setspace, harvard, perpage, and url. They are downloaded from the Internet free of charge.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

> From 1973 to 1977, the main tool applicable to the study of countable nonstandard models of arithmetic was the back-and-forth construction. Fashions change, however, and nowadays the preference is for the flashier use of indicators. A notable exception to this trend is in the study of recursively saturated models of arithmetic. The facts that recursive saturation is defined in terms of the realisability of "nice" types and that the back-and-forth technique relies for its application on the realisability of "nice" types practically dictate a continued prominent rôle for back-and-forth constructions in this area of study.
>
> Craig Smoryński (1982a)
> Back-and-forth inside a recursively saturated model of arithmetic

There had been a huge amount of research in the late 1970s and early 1980s on *indicators* since they were first formulated by Jeff Paris and Laury Kirby (Kirby and Paris 1977, Kirby 1977) and independence results were obtained from them. Indicators then ceased to be popular. They were only picked up again recently by Richard Kaye (2008) recently, who looked at them from a new point of view.

In Kaye (2008), which we refer to as GCMA for convenience, a model of arithmetic together with its initial segments were considered as a *single* topological space. An indicator serves to give a very rough idea of distance. From this more global point of view, Kaye discovered, using techniques in topology, a family of cuts called the *generic cuts* having various nice properties.

The first aim of this thesis is to generalize this. In Chapter 2, we give the basic definitions in the thesis, namely that of a *notion of intervals*, and that of a *species*. A notion of intervals is an attempt to extract the topology from an indicator. A species is essentially a set of cuts that can be indicated. The main relaxation in the definitions is using *class* functions in the sense in arithmetic, instead of definable functions. The aim of Chapter 2 is to verify that these new notions are in fact just other faces of indicators. Chapter 3 is devoted to showing that notions of intervals and species represent the same concept.

In Chapters 4, 5 and 6, we show that most results in GCMA remain true in this more general situation. Chapter 4 sets up the topology in which we will work. The major step there is proving any complete species in a countable model is homeomorphic to the Cantor set. This enables us to apply the Baire Category Theorem to, and play Banach–Mazur games on, our space to obtain information about *enforceable* subsets. The main aim of Chapter 5 is showing the existence of generic cuts in our context. On the way, we define and prove the existence of what we call *pregeneric intervals*, which combine small and constant intervals in GCMA. A detailed breakdown proof is given so that more information is squeezed out. In Chapter 6, we start to study how generic cuts behave under the action of the automorphism group of the model. The back-and-forth system that we took from GCMA is what most our results there are based on.

A side product of these chapters is some results that tell us how countable arithmetically saturated models of Peano arithmetic (PA) look like. It is well-known that such models are very homogeneous, both in the technical sense and in the non-technical sense. We provide some ways to describe this homogeneity more precisely.

We move from generalizing to properly extending the results of GCMA in Chapter 7. The main theorem there is a syntactic criterion for conjugacy of generic cuts. A nice corollary of this is a quantifier elimination result, which says that if I is a generic cut in a model M of PA, then the orbit of an element in (M, I) under the action of $\mathrm{Aut}(M, I)$ is completely determined by classes that are relatively low in the formula hierarchy. Apart from suggesting the study of such automorphism groups will be exciting, these theorems also lay firm ground work for future research in this area.

We gather together various facts about *elementary generic cuts* in Chapter 8, and survey their relationships to the elementary cuts that appeared in the literature. In particular, we show that elementary generic cuts give new examples of *free cuts*, a notion introduced by Roman Kossak (1986, 1995). This partially answers his question on the cardinality of orbits of free cuts, and possibly supplies new ideas in tackling his other problems too.

This thesis concludes with a discussion on some interesting topics that are worth further investigation. This includes some preliminary results on the automorphism group of (M, I) where I is a generic cut in a model M.

The notation used in this thesis is standard. We had intended the materials covered to be self-contained, but we failed. The reader may find the book by Kaye (1991) and the book by Kossak and Schmerl (2006) helpful. The view we take on coding in arithmetic is as in the paper by Kaye and Wong (2007). Familiarity with Kaye's GCMA (Kaye 2008) is essential. We assume some knowledge in semiregular, regular and strong cuts, the basic properties of which can be found in Kirby and Paris (1977). Oxtoby (1971) contains some definitions and proofs that we have omitted in Chapter 4.

Global assumptions are gathered at the beginning of each chapter, except possibly in the final one.

CHAPTER 2

THE SETTING

> Mathematical discovery is by no means a matter of systematic deductive procedure. It involves insight, imagination, and long explorations along paths that sometimes lead nowhere. Axiomatic presentations serve to describe and communicate the fruits of this activity, often in a different order to that in which they were arrived at. They lend a coherence and unity to their subject matter, an overview of its extent and limitations.
>
> Robert Goldblatt (1984)
> **Topoi**, §1.2, p. 14

Throughout this thesis, M is a nonstandard model of PA. We write $\mathscr{L}_{\mathrm{A}}$ for the usual first order language $\{+, \times, <, 0, 1\}$ for arithmetic, and $\langle \cdot, \cdot \rangle$ for the standard pairing function in $\mathscr{L}_{\mathrm{A}}$. Let $\mathrm{cl}(\bar{c})$ denote the definable closure of the finite tuple $\bar{c} \in M$. The quantifier "there exist cofinally many" is denoted by Q.

By M^*, we mean the set $M \cup \{\infty\}$, where ∞ is a new element that we use to represent a point at infinity. So by convention, we have $x < \infty$ and $\infty - x = \infty$ for every $x \in M$. For notational convenience, we use $\overset{*}{\forall}$ and $\overset{*}{\exists}$ for quantification over M^*. If $B \in M$, then $M_{<B}$ and $M_{\leqslant B}$ denote respectively the coded sets

$$\{x \in M : x < B\} \text{ and } \{x \in M : x \leqslant B\}.$$

A *cut* in M is a nonempty initial segment closed under successors. We write $I \subseteq_{\mathrm{e}} M$ to mean "I is a cut in M." Unlike in GCMA, we do not require cuts to be $\mathscr{L}_{\mathrm{A}}$ structures.

For $a, b \in M^*$, we denote the set

$$\{x \in M^* : a \leqslant x \leqslant b\}$$

by $\lceil a, b \rfloor$. Define

$$\mathcal{C} = \{I : I \subseteq_{\mathrm{e}} M\} \text{ and } \mathcal{S} = \{\lceil a, b \rfloor : a, b \in M^*\}.$$

Elements of $\mathcal{S}$ are called *semi-intervals*. We say that a semi-interval is *finite* if and only if both its endpoints are in M.

For a cut I and and a semi-interval $\lceil a, b \rfloor$, we write $I \in \lceil a, b \rfloor$ to mean $a \in I < b$. Given $\mathcal{Z} \subseteq \mathcal{C}$ and $\lceil a, b \rfloor \in \mathcal{S}$, define

$$\lceil a, b \rfloor_{\mathcal{Z}} = \{I \in \mathcal{Z} : a \in I < b\}.$$

When the set $\mathcal{Z} \subseteq \mathcal{C}$ is clear from the context, we sometimes identify a semi-interval $\lceil a, b \rfloor$ with the set $\lceil a, b \rfloor_{\mathcal{Z}}$.

The automorphism group of M is denoted by $\mathrm{Aut}(M)$. All actions by automorphisms are written on the right. For instance, if $x \in M$, then the image of x under g is written as xg. If $I \in \mathcal{C}$ and $\lceil a, b \rfloor \in \mathcal{S}$, then let I^g and $\lceil a, b \rfloor^g$ be respectively the image of I and the image of $\lceil a, b \rfloor$ under the automorphism $g \in \mathrm{Aut}(M)$. Note that the action of $\mathrm{Aut}(M)$ extends naturally to one on M^*. If $\bar{c} \in M$, then $\mathrm{Aut}(M, \bar{c})$ denotes the pointwise stabilizer of $\bar{c}$ in $\mathrm{Aut}(M)$. Similarly, $\mathrm{Aut}(M, I)$ denotes the setwise stabilizer of I in $\mathrm{Aut}(M)$ if I is a cut in M.

We first set the scene by seeking a generalization of the definition of indicators in GCMA. A different perspective is employed here. Although the notion of an *interval* is very important in the study of generic cuts, it is only touched on very briefly in GCMA. Keeping in mind that generic cuts are our main interest in the long run, we attempt to isolate this notion.

Definition. A subclass $\mathcal{B} \subseteq \mathcal{S}$ is a *notion of intervals* if and only if

(0) $\mathcal{B}$ is nonempty;

(1) $\mathcal{B}$ is invariant under the action of $\mathrm{Aut}(M)$;

(2) $\forall \lceil a, b \rfloor \in \mathcal{B}\ \exists c \in M\ (a < c < b \text{ and } \lceil a, c \rfloor, \lceil c, b \rfloor \in \mathcal{B})$;

(3) $\forall \lceil a, b \rfloor \in \mathcal{B}\ \forall \lceil u, v \rfloor \in \mathcal{S}\ (\lceil a, b \rfloor \subseteq \lceil u, v \rfloor \Rightarrow \lceil u, v \rfloor \in \mathcal{B})$; and

(4) for every $B \in M$, there exists a recursive Σ_1 type $p(x, y)$ over M, possibly with parameters from M, such that

$$\forall a, b < B\ \left(\lceil a, b \rfloor \in \mathcal{B} \Leftrightarrow M \models \bigwedge\!\!\!\bigwedge p(a, b)\right).$$

If $\mathcal{B}$ is a notion of intervals and $\lceil a, b \rfloor \in \mathcal{B}$, then we say that $\lceil a, b \rfloor$ is a $\mathcal{B}$*-interval* and write $a \ll_{\mathcal{B}} b$ for it. Sometimes, we refer to a $\mathcal{B}$-interval $\lceil a, b \rfloor$ as $[a, b]$ for emphasis. When there is no risk of ambiguity, references to $\mathcal{B}$ are often omitted.

Remark. By convention, all types that appear in this thesis may contain finitely many parameters from M, and these parameters are not necessarily all indicated.

At first sight, it is not entirely obvious that this definition generalizes that of an indicator. The following will make the relationship clear.

Definition. Let $B \in M$ and $\mathcal{B} \subseteq \mathcal{S}$. A function $Y\colon M_{<B} \times M_{<B} \to M$ is an *indicator for* $\mathcal{B}$ *below* B if and only if it is coded in M, and

$$\forall a, b < B\ (\lceil a, b \rfloor \in \mathcal{B} \Leftrightarrow Y(a, b) > \mathbb{N}).$$

An indicator Y for $\mathcal{B}$ below B is *monotone* if and only if

$$\forall a, b, u, v \leqslant B\ \big((a \leqslant u \wedge v \leqslant b) \to Y(a, b) \geqslant Y(u, v)\big).$$

Proposition 2.1. Let $\mathcal{B} \subseteq \mathcal{S}$. The following are equivalent:

(1) $\mathcal{B}$ satisfies axiom (4) for a notion of intervals;

(2) $\mathcal{B}$ has an indicator below B for any $B \in M$;

(3) $\mathcal{B}$ has a monotone indicator below B for any $B \in M$.

Proof. Fix $\mathcal{B} \subseteq \mathcal{S}$. We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

To prove (1) $\Rightarrow$ (2), suppose (1) holds. Pick $B \in M$. Using axiom (4) for a notion of intervals, let $p(x, y)$ be a recursive Σ_1 type over M such that

$$\forall a, b < B \ \Big(\lceil a, b \rfloor \in \mathcal{B} \Leftrightarrow M \models \bigwedge\!\!\!\bigwedge p(a, b)\Big) .$$

Let $\bar{d} \in M$ be the parameters that appear in $p(x, y)$, and write $p(x, y)$ as $p(x, y, \bar{d})$. Then $p(x, y, \bar{d})$ is coded in M (by Corollary 3.7 in Kaye (1991), say). Let c be a code for $p(x, y, \bar{d})$ in M, i.e., c is an element of M such that

$$\{(c)_n : n \in \mathbb{N}\} = \{\ulcorner \phi(x, y, \bar{z}) \urcorner : \phi(x, y, \bar{d}) \in p(x, y, \bar{d})\}.$$

Define a function $Y \colon M_{<B} \times M_{<B} \to M$ by setting

$$Y(x, y) = (\mu n)\big(\neg \operatorname{Sat}_{\Sigma_1}((c)_n, [x, y, \bar{d}])\big)$$

for all $x, y < B$. It is easy to check that Y is coded in M, and for all $x, y < B$,

$$\begin{aligned} \lceil x, y \rfloor \in \mathcal{B} \ \text{ iff } \ & M \models \bigwedge\!\!\!\bigwedge p(x, y, \bar{d}) && \text{by our choice of } p(x, y, \bar{d}), \\ \text{ iff } \ & Y(x, y) > \mathbb{N} && \text{by the definition of } Y. \end{aligned}$$

So Y is an indicator for $\mathcal{B}$ below B.

If $B \in M$ and Y be an indicator for $\mathcal{B}$ below B, then one can readily see that the function $Y'\colon M_{<B} \times M_{<B} \to M$ defined by

$$\forall x, y < B \ \big(Y'(x,y) = \max\{Y(a,b) : a, b \in \lceil x, y \rfloor\}\big)$$

is a monotone indicator for $\mathcal{B}$ below B. This proves (2) $\Rightarrow$ (3).

Let $B \in M$ and Y be a monotone indicator for $\mathcal{B}$ below B. Consider the type

$$p(x,y) = \{Y(x,y) > n : n \in \mathbb{N}\},$$

which is recursive and Σ_1. For all $x, y < B$, we have

$$\begin{aligned} \lceil x, y \rfloor \in \mathcal{B} \ \text{ iff } \ & Y(x,y) > \mathbb{N} && \text{by the choice of } Y, \\ \text{iff } \ & M \models \textstyle\bigwedge\!\!\!\bigwedge p(x,y) && \text{by the definition of } p(x,y). \end{aligned}$$

Since the choice of $B \in M$ is arbitrary, we conclude that axiom (4) holds for $\mathcal{B}$. □

The generalized definition gives rise to some important new examples, but let us look at the old one first.

Example 2.2. Let Y be an indicator in the sense of GCMA. We always assume such an indicator satisfy

$$M \models \exists x \exists y \, Y(x,y) \geqslant n$$

for every $n \in \mathbb{N}$ to avoid triviality. We call indicators in this old setting *GCMA indicator* in this thesis. Set

$$\begin{aligned} \mathcal{B}^Y = {} & \{\lceil a, b \rfloor \in \mathcal{S} : Y(a,b) > \mathbb{N}\} \\ & \cup \{\lceil a, \infty \rfloor \in \mathcal{S} : \exists b \in M \ \ Y(a,b) > \mathbb{N}\}. \end{aligned}$$

It can be verified that $\mathcal{B}^Y$ is a notion of intervals. Actually, the definition of a notion of intervals here closely resembles that of an indicator in GCMA. It would be nice if one can find a formulation that does not even remotely involve an indicator-like function.

Example 2.3. Fix a recursive sequence $(t_n(x))_{n\in\mathbb{N}}$ of $\mathscr{L}_{\mathrm{A}}$ Skolem functions with the following properties:

- $\forall n \in \mathbb{N}\, \forall x \in M\ (t_n(x) < t_{n+1}(x))$;
- $\forall n \in \mathbb{N}\, \forall x \in M\ (x < t_n(x) \leqslant t_n(x+1))$; and
- for every $\mathscr{L}_{\mathrm{A}}$ Skolem function $s(x)$, there is an $n \in \mathbb{N}$ such that for all $x \in M$, we have $s(x) < t_n(x)$.

If M is recursively saturated, then the set

$$\mathcal{B}^{\mathrm{elem}} = \{\lceil a, b\rfloor \in \mathcal{S} : \forall n \in \mathbb{N}\ (t_n(a) < b)\}$$

is a notion of intervals. Intervals in $\mathcal{B}^{\mathrm{elem}}$ are sometimes called *elementary intervals*. By a diagonalization argument, it can be seen that there is no definable function $Y\colon M^2 \to M$ satisfying

$$Y(a, b) > \mathbb{N} \ \text{ iff } \ [a, b] \in \mathcal{B}^{\mathrm{elem}}$$

for all $a, b \in M$. (Alternatively, use Lemma 3.4.) Therefore, our definition of a notion of intervals is strictly more general than its counterpart in GCMA.

These two are the key examples that will appear over and over again in this thesis. The following is one that we do not know much about yet.

Example 2.4. Let Sat be a partial inductive satisfaction class for M. Find a recursive sequence of $(t_n^{\mathrm{Sat}}(x))_{n\in\mathbb{N}}$ with the monotonicity and domination properties as in the previous example, but in the language $\mathscr{L}_{\mathrm{A}} \cup \{\mathrm{Sat}\}$. Then the set

$$\mathcal{B}^{\mathrm{Sat}} = \{\lceil a, b\rfloor^g \in \mathcal{S} : g \in \mathrm{Aut}(M) \text{ and } \forall n \in \mathbb{N}\ (t_n^{\mathrm{Sat}}(a) < b)\}$$

is again a notion of intervals, provided (M, Sat) is sufficiently saturated.

Question 2.5. Note that the notions of intervals above form a nice hierarchy, namely,

$$\mathcal{B}^Y \supsetneq \mathcal{B}^{\text{elem}} \supsetneq \mathcal{B}^{\text{Sat}}$$

where Y and Sat are as in the previous examples. Are there two notions of intervals $\mathcal{B}_1, \mathcal{B}_2$ such that $\mathcal{B}_1 \not\subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \not\subseteq \mathcal{B}_1$?

Many facts about indicators remain true in our setting.

Fact 2.6. Let $\mathcal{B}$ be a notion of intervals.

(a) Every $\mathcal{B}$-interval contains a proper subinterval.

(b) If $[a,b] \in \mathcal{B}$, then $b - a > \mathbb{N}$.

(c) $\forall a, b \in M^* \ (a \ll b \Rightarrow a < b)$.

(d) $\forall a, b, c \in M^* \ (a \leqslant b \ll c \ \text{ or } \ a \ll b \leqslant c \Rightarrow a \ll c)$.

(e) $\forall [a,b] \in \mathcal{B} \ \forall c \in M \ (\lceil a, c \rfloor \in \mathcal{B} \ \text{ or } \ \lceil c, b \rfloor \in \mathcal{B})$. □

In general, indicators in the current setting may not have the intermediate value property as those in GCMA. Fortunately, a weak form of it is sufficient for our purposes. The following lemma is formulated in terms of the standard cut because that is the place where we are mostly interested in. Actually, it is true of other cuts as well.

Lemma 2.7. Let $\mathcal{B}$ be a notion of intervals, $B \in M$ and $Y \in M$ be an indicator for $\mathcal{B}$ below B. If $[a,b] \subseteq M_{<B}$ is a $\mathcal{B}$-interval, then

$$\{n > \mathbb{N} : M \models \exists \lceil u, v \rfloor \subseteq [a,b] \ (Y(u,v) = n)\} \subseteq_{\text{dcf}} M \setminus \mathbb{N}.$$

Proof. Let $\mathcal{B}$ be a notion of intervals, $B \in M$ and $Y \in M$ be an indicator for $\mathcal{B}$ below B. Take a $\mathcal{B}$-interval $[a,b] \subseteq M_{<B}$ and define X to be the set

$$\{n \in \mathbb{N} : \exists \lceil u, v \rfloor \subseteq [a,b] \ (Y(u,v) = n)\}.$$

Note that by X is nonempty by Fact 2.6(b).

Suppose $X \not\subseteq_{\mathrm{cf}} \mathbb{N}$. Then X has an upper bound in $\mathbb{N}$, say D. Now for every $x \in [a, b]$,

$$\begin{aligned} \lceil x, b \rfloor \in \mathcal{B} \ \text{ iff } \ & Y(x, b) > \mathbb{N} && \text{since } Y \text{ is an indicator for } \mathcal{B} \text{ below } B, \\ \text{iff } \ & Y(x, b) > D && \text{by our choice of } D \text{ and axiom (3) for intervals.} \end{aligned}$$

Therefore, since the set $\{x \in [a, b] : \lceil x, b \rfloor \in \mathcal{B}\}$ contains a and is bounded above by b, it has a maximum element, say $x^* \in M$. In particular, $\lceil x^*, b \rfloor \in \mathcal{B}$ but $\lceil x^* + 1, b \rfloor \notin \mathcal{B}$. Thus $\lceil x^*, x^* + 1 \rfloor \in \mathcal{B}$ by Fact 2.6(e). This contradicts Fact 2.6(b).

Therefore, $X \subseteq_{\mathrm{cf}} \mathbb{N}$. By overspill, we get

$$\{n > \mathbb{N} : M \models \exists \lceil u, v \rfloor \subseteq [a, b]\ (Y(u, v) = n)\} \subseteq_{\mathrm{dcf}} M \setminus \mathbb{N},$$

as required. $\square$

From another point of view, we follow up one of the objectives in GCMA and attempt to seek a characterization of properties of cuts that has indicators in the sense of Kirby and Paris (1977).

Definition. A class $\mathscr{Z} \subseteq \mathcal{C}$ is a *species* if and only if

(0) $\mathscr{Z}$ is nonempty;

(1) $\mathscr{Z}$ is invariant under the action of $\mathrm{Aut}(M)$;

(2) if $a, b \in M^*$ and $I \in \mathscr{Z}$ such that $a \in I < b$, then there is a cut $J \in \mathscr{Z}$ with $a \in J < b$ that is distinct from I; and

(4) for every $B \in M$, there exists a recursive Σ_1 type $p(x, y)$ over M, possibly with parameters from M, such that

$$\forall a, b < B\ \left(\exists I \in \mathscr{Z}\ (a \in I < b) \Leftrightarrow M \models \bigwedge\!\!\!\bigwedge p(a, b)\right).$$

If I is an element of $\mathcal{Z} \subseteq \mathcal{C}$, then we say that I is a *$\mathcal{Z}$-cut*.

Similar to notions of intervals, the type condition in the above definition corresponds to the existence of indicators. The proof is almost identical.

Definition. Let $B \in M$ and $\mathcal{Z} \subseteq \mathcal{C}$. A function $Y\colon M_{<B} \times M_{<B} \to M$ is an *indicator for $\mathcal{Z}$ below B* if and only if it is coded in M, and

$$\forall a, b < B\ (\exists I \in \mathcal{Z}\ (a \in I < b) \Leftrightarrow Y(a,b) > \mathbb{N}).$$

An indicator Y for $\mathcal{Z}$ below B is *monotone* if and only if

$$\forall a, b, u, v \leqslant B\ \big((a \leqslant u \wedge v \leqslant b) \to Y(a,b) \geqslant Y(u,v)\big).$$

Proposition 2.8. Let $\mathcal{Z} \subseteq \mathcal{C}$. The following are equivalent:

(1) $\mathcal{Z}$ satisfies axiom (4) for a species;

(2) $\mathcal{Z}$ has an indicator below B for any $B \in M$;

(3) $\mathcal{Z}$ has a monotone indicator below B for any $B \in M$. □

It is interesting to see exactly how much stronger this type condition is when compared with the original definability requirement in GCMA.

Question 2.9. Fix a species $\mathcal{Z}$. Is there always a function $Y\colon M^2 \to M$ such that

- $\forall x, y \in M\ (\exists I \in \mathcal{Z}\ (x \in I < y) \Leftrightarrow Y(x,y) > \mathbb{N})$, and
- for every $B \in M$, the set $\{\langle x, y, Y(x,y)\rangle : x, y \leqslant B\}$ is coded in M?

We have essentially the same examples as of notions of intervals.

Example 2.10. Let Y be a GCMA indicator, and

$$\mathcal{Z}^Y = \{I \subseteq_{\mathrm{e}} M : \forall n \in \mathbb{N}\ \forall x \in I\ \exists y \in I\ Y(x,y) \geqslant n\}$$

be the topological space of cuts that Kaye introduced in GCMA. Then many natural dense subset of $\mathscr{Z}^Y$ are species. For example, families of cuts in the Paris–Kirby hierarchy (Kirby 1977, Paris 1980) are all species, and so are collections of cuts satisfying a recursively axiomatized theory (cf. Theorem 5.1 in Kirby (1977) and Theorem 14 in Kirby, McAloon and Murawski (1981)).

Example 2.11. Suppose M is recursively saturated. Let $(t_n)_{n\in\mathbb{N}}$ be the sequence of Skolem functions chosen in Example 2.3, and $\mathscr{Z}^{\mathrm{elem}}$ be the set of cuts that are closed under t_n for each $n \in \mathbb{N}$, i.e.,

$$\mathscr{Z}^{\mathrm{elem}} = \{I \subseteq_{\mathrm{e}} M : \forall n \in \mathbb{N}\, \forall x \in I\ t_n(x) \in I\}.$$

Then $\mathscr{Z}^{\mathrm{elem}}$ is a species. Cuts in $\mathscr{Z}^{\mathrm{elem}}$ are often called *elementary cuts*, and we write $I \prec_{\mathrm{e}} M$ to mean $I \in \mathscr{B}^{\mathrm{elem}}$. Chapter 8 is devoted to these cuts.

Similarly, if Sat is a partial inductive satisfaction class, (M, Sat) is sufficiently saturated, and $(t_n^{\mathrm{Sat}})_{n\in\mathbb{N}}$ is the sequence of functions chosen in Example 2.4, then the set

$$\mathscr{Z}^{\mathrm{Sat}} = \{I^g \subseteq_{\mathrm{e}} M : g \in \mathrm{Aut}(M) \text{ and } \forall n \in \mathbb{N}\, \forall x \in I\ t_n(x) \in I\}$$

is again a species.

CHAPTER 3

CONNECTING INTERVALS AND SPECIES

> It seems that in mathematics there are sometimes two or more ways of proving the same result. This is often mysterious, and seems to go against the grain, for we often have a deep-down feeling that if we choose the 'right' ideas or definitions, there must be only one 'correct' proof. [...]
>
> Sometimes this mystery can be resolved by analysing the apparently different proofs into their fundamental ideas. It often turns out that, 'underneath the bonnet', there is actually just one key mathematical concept, and two seemingly different arguments are in some sense 'the same'. But sometimes there really are two different approaches to a problem. This should not be disturbing, but should instead be seen as a great opportunity. After all, two approaches to the same idea indicates that there are some new mathematics to be investigated and some new connections to be found and exploited, which hopefully will uncover a wealth of new results.
>
> Richard Kaye (2007)
> **The Mathematics of Logic**, §1.1, p. 1

As we saw in the previous chapter, notions of intervals and species are just alternative ways to describe indicators. In this chapter, we develop a natural correspondence between the two concepts. Most of the materials here are generalizations of results from GCMA.

Definition. For a notion of intervals $\mathcal{B}$, define

$$\mathscr{Z}_{\mathcal{B}} = \{I \in \mathcal{C} : \overset{*}{\forall} a \in I \ \overset{*}{\forall} b > I \ \lceil a, b \rfloor \in \mathcal{B}\}.$$

For a species $\mathcal{Z}$, define

$$\mathscr{B}_{\mathcal{Z}} = \{\lceil a, b \rfloor \in \mathcal{S} : \exists I \in \mathcal{Z} \ (a \in I < b)\}.$$

Morally, $\mathscr{Z}$ and $\mathscr{B}$ are the maps we need for the correspondence. The idea is that notions of intervals are like *habitats* and species are like *populations* of a particular group of organisms. The habitat that a population $\mathcal{Z}$ occupies is exactly $\mathscr{B}_{\mathcal{Z}}$. However, it is possible that different populations share the same habitat. Such populations are said to be *symbiotic*. The set $\mathscr{Z}_{\mathcal{B}}$ is the collection of all organisms that a habitat $\mathcal{B}$ can accommodate. It is a population that includes all the smaller populations that can be considered individually in this habitat. Thus, in a sense, one can identify a habitat with the class of populations that lives there.

Please be very careful when one attempts to push this analogy any further.

To put all these into proper mathematics, we first look at some special cuts with respect to a notion of intervals. They help us understand the $\mathscr{Z}$ map.

Definition. Let $\mathcal{B}$ be a notion of intervals, and $a, b \in M^*$. Define

$$M_{\mathcal{B}}(a) = \inf\{y \in M : \lceil a, y \rfloor \in \mathcal{B}\}, \text{ and}$$
$$M_{\mathcal{B}}[b] = \sup\{x \in M : \lceil x, b \rfloor \in \mathcal{B}\}.$$

We say that $M_{\mathcal{B}}(a)$ *exists* if and only if

$$\exists y \in M \ \lceil a, y \rfloor \in \mathcal{B}.$$

Similarly, $M_{\mathcal{B}}[b]$ *exists* if and only if

$$\exists x \in M \ \lceil x, b \rfloor \in \mathcal{B}.$$

Whenever there is no risk of ambiguity, the subscripts are dropped.

Proposition 3.1. Let $\mathcal{B}$ be a notion of intervals and $a, b \in M^*$.

(a) If $M_{\mathcal{B}}(a)$ exists, then it is the smallest cut in $\mathscr{Z}_{\mathcal{B}}$ that contains a.

(b) If $M_{\mathcal{B}}[b]$ exists, then it is the biggest cut in $\mathscr{Z}_{\mathcal{B}}$ that does not contain b.

In particular, $\mathscr{L}_\mathcal{B}$ has a minimum and a maximum, namely $M_\mathcal{B}(0)$ and $M_\mathcal{B}[\infty]$ respectively.

Proof. Let $\mathcal{B}$ be a notion of intervals, and $a, b \in M^*$. We prove the part for $M_\mathcal{B}(a)$ only. The proof for $M_\mathcal{B}[b]$ is similar. Suppose $M(a)$ exists.

Note that if $a > M(a)$, then $\lceil a, a \rfloor \in \mathcal{B}$ by the definition of $M(a)$, contradicting Fact 2.6(b). So $a \in M(a)$.

Let $x, y \in M^*$ such that $x \in M(a) < y$. By the definition of $M(a)$, we know that

$$\lceil a, x \rfloor \notin \mathcal{B} \text{ and } \lceil a, y \rfloor \in \mathcal{B}.$$

By Fact 2.6(e), we must have $\lceil x, y \rfloor \in \mathcal{B}$ as required. In particular, $M(a)$ is closed under successors by Fact 2.6(b).

It remains to prove that $M(a)$ is the smallest cut with the required properties. If I is a $\mathscr{L}_\mathcal{B}$-cut containing a, then by the definition of $\mathscr{L}_\mathcal{B}$,

$$\forall y > I \ \lceil a, y \rfloor \in \mathcal{B},$$

and so $M(a) \subseteq_\mathrm{e} I$ by the minimality of $M(a)$. □

These cuts are well-known in the context of GCMA and in $\mathcal{B}^\mathrm{elem}$.

Example 3.2. Let Y be a GCMA indicator. Results in GCMA show that

$$M_{\mathcal{B}^Y}(a) = \sup\{(\mu y)(Y(a, y) \geqslant n) : n \in \mathbb{N}\}, \text{ and}$$
$$M_{\mathcal{B}^Y}[b] = \inf\{(\mu x)(Y(x, b) < n) : n \in \mathbb{N}\}$$

for all $a, b \in M$.

Example 3.3. If $a \in M$, then $M_{\mathcal{B}^\mathrm{elem}}(a)$ is the smallest elementary initial segment of M containing a, and we denote this by $\overline{\mathrm{cl}}(a)$. Standard results about nonstandard models of PA tell us that

$$a \in \mathrm{cl}(a) \prec_\mathrm{cf} \overline{\mathrm{cl}}(a) \prec_\mathrm{e} M$$

for all $a \in M$. These cuts always exist.

As a first application, we show that the notion of elementary intervals does not correspond to GCMA indicator.

Lemma 3.4. Let Y be a GCMA indicator. Then $M_{\mathcal{B}^Y}(a) \neq M_{\mathcal{B}^{\mathrm{elem}}}(x)$ for all $x \in M$ and nonstandard $a \in M$ when $M_{\mathcal{B}^Y}(a)$ and $M_{\mathcal{B}^{\mathrm{elem}}}(x)$ both exist. Moreover, if $M \not\models \mathrm{Th}(\mathbb{N})$, then we can allow a to be standard.

Proof. Fix a GCMA indicator Y. Pick $a, x \in M$ such that $M_{\mathcal{B}^Y}(a)$ and $M_{\mathcal{B}^{\mathrm{elem}}}(x)$ both exist and a is nonstandard. By Proposition 3.1, if $a > M_{\mathcal{B}^{\mathrm{elem}}}(x)$, then

$$M_{\mathcal{B}^{\mathrm{elem}}}(x) < a \in M_{\mathcal{B}^Y}(a)$$

and we are done. So suppose $a \in M_{\mathcal{B}^{\mathrm{elem}}}(x)$. Then since a is nonstandard, we get

$$\begin{aligned} M_{\mathcal{B}^Y}(a) &= \sup\{(\mu y)(Y(a,y) \geqslant n) : n \in \mathbb{N}\} \\ &< (\mu y)(Y(a,y) \geqslant a) \in \mathrm{cl}(a) \subseteq M_{\mathcal{B}^{\mathrm{elem}}}(a) \subseteq_{\mathrm{e}} M_{\mathcal{B}^{\mathrm{elem}}}(x) \end{aligned}$$

by Proposition 3.1, which is what we want.

For the "moreover" part, suppose $M \not\models \mathrm{Th}(\mathbb{N})$. Pick a nonstandard $\nu \in \mathrm{cl}(\varnothing)$ and a natural number $a \in \mathbb{N}$. Then for every $x \in M$ such that $M_{\mathcal{B}^{\mathrm{elem}}}(x)$ exists, we have

$$\begin{aligned} M_{\mathcal{B}^Y}(a) &= \sup\{(\mu y)(Y(a,y) \geqslant n) : n \in \mathbb{N}\} \\ &< (\mu y)(Y(a,y) \geqslant \nu) \in \mathrm{cl}(\varnothing) \subseteq M_{\mathcal{B}^{\mathrm{elem}}}(0) \subseteq_{\mathrm{e}} M_{\mathcal{B}^{\mathrm{elem}}}(x) \end{aligned}$$

by Proposition 3.1, as required. □

Let us return to studying general notions of intervals. Our next lemma says that cuts of the form $M(a)$ and cuts of the form $M[b]$ are intrinsically quite different because of axiom (2).

Lemma 3.5. Let $\mathcal{B}$ be a notion of intervals. If $a, b \in M^*$ such that both $M_\mathcal{B}(a)$ and $M_\mathcal{B}[b]$ exist, then $M_\mathcal{B}(a) \neq M_\mathcal{B}[b]$.

Proof. Let $\mathcal{B}$ be a notion of intervals and $a, b \in M^*$ such that $M_\mathcal{B}(a)$ and $M_\mathcal{B}[b]$ exist. Suppose $M_\mathcal{B}(a) = M_\mathcal{B}[b]$. By Proposition 3.1, we know that $M_\mathcal{B}(a) \in \mathscr{Z}_\mathcal{B}$ and

$$a \in M_\mathcal{B}(a) = M_\mathcal{B}[b] < b.$$

So $\lceil a, b \rfloor \in \mathcal{B}$ by the definition of $\mathscr{Z}_\mathcal{B}$. Using axiom (2) for notions of intervals, let $c \in \lceil a, b \rfloor$ such that $a \ll_\mathcal{B} c \ll_\mathcal{B} b$. Now, either $c \in M_\mathcal{B}(a)$ or $M_\mathcal{B}[b] < c$. The former contradicts the minimality of $M_\mathcal{B}(a)$, while the latter contradicts the maximality of $M_\mathcal{B}[b]$. $\square$

Remark. Note that by convention, $\inf \varnothing = M$ and $\sup \varnothing = \varnothing$. It can thus be checked that the only situation in which elements $a, b \in M^*$ and a notion of intervals $\mathcal{B}$ can satisfy $M_\mathcal{B}(a) = M_\mathcal{B}[b]$ is when $a = b = \infty$ and $M_\mathcal{B}[b] = M$. However, we usually do not consider situations when $M(a)$ or $M[b]$ does not exist.

These enable us to show that $\mathscr{Z}$ is a well-defined map from the class of all notions of intervals to the class of all species.

Proposition 3.6. If $\mathcal{B}$ is a notion of intervals, then $\mathscr{Z}_\mathcal{B}$ is the biggest species $\mathcal{Z}$ satisfying

(i) $\forall I \in \mathcal{Z}\ \forall a, b \in M^*\ (a \in I < b \Rightarrow \lceil a, b \rfloor \in \mathcal{B})$; and

(ii) $\forall [a, b] \in \mathcal{B}\ \exists I \in \mathcal{Z}\ (a \in I < b)$.

Proof. Let $\mathcal{B}$ be a notion of intervals. Clause (i) is immediate from the definition of $\mathscr{Z}_\mathcal{B}$. To prove (ii) for $\mathscr{Z}_\mathcal{B}$, let $[a, b] \in \mathcal{B}$. By Proposition 3.1, $a \in M(a) \in \mathscr{Z}_\mathcal{B}$, and by the minimality of $M(a)$, we have $M(a) < b$. In particular, since $\mathcal{B}$ is nonempty, so is $\mathscr{Z}_\mathcal{B}$.

Next, we verify that $\mathscr{Z}_\mathcal{B}$ is a species. For axiom (1), pick $I \in \mathscr{Z}_\mathcal{B}$ and $g \in \mathrm{Aut}(M)$. We need to show that $I^g \in \mathscr{Z}_\mathcal{B}$. Let $a, b \in M$ such that $a \in I^g < b$. Then $ag^{-1} \in I < bg^{-1}$ and so since $I \in \mathscr{Z}_\mathcal{B}$, we get $\lceil ag^{-1}, bg^{-1} \rfloor \in \mathcal{B}$. Axiom (1) for intervals says that $\mathcal{B}$ is closed under automorphisms of M. In particular,

$$\lceil a,b\rfloor = \lceil ag^{-1}g, bg^{-1}g\rfloor = \lceil ag^{-1}, bg^{-1}\rfloor^{g} \in \mathcal{B},$$

proving axiom (1) for species for $\mathscr{Z}_{\mathcal{B}}$.

To prove axiom (2), let $a,b \in M^*$ and $I \in \mathscr{Z}_{\mathcal{B}}$ such that $a \in I < b$. Then $\lceil a,b\rfloor \in \mathcal{B}$ by the definition of $\mathscr{Z}_{\mathcal{B}}$. Using axiom (2) for intervals, let $c \in M$ such that

$$a < c < b \;\text{ and }\; \lceil a,c\rfloor, \lceil c,b\rfloor \in \mathcal{B}.$$

Recall that we want to find a $\mathscr{Z}_{\mathcal{B}}$-cut in $\lceil a,b\rfloor$ that is distinct from I. If $c \in I$, then $M_{\mathcal{B}}[c]$ gives us what we want; otherwise, take $M_{\mathcal{B}}(c)$.

For axiom (4), let $B \in M$ be arbitrary, and $p(x,y)$ be a recursive Σ_1 type such that

$$\forall a,b < B \;\left(\lceil a,b\rfloor \in \mathcal{B} \Leftrightarrow M \models \bigwedge\!\!\!\bigwedge p(a,b)\right).$$

We show that

$$\forall a,b < B \;\left(\exists I \in \mathscr{Z}_{\mathcal{B}}\ (a \in I < b) \Leftrightarrow M \models \bigwedge\!\!\!\bigwedge p(a,b)\right).$$

Fix $a,b < B$. Suppose $I \in \mathscr{Z}_{\mathcal{B}}$ such that $a \in I < b$. By the definition of $\mathscr{Z}_{\mathcal{B}}$, we have $\lceil a,b\rfloor \in \mathcal{B}$ and so $M \models \bigwedge\!\!\!\bigwedge p(a,b)$. Conversely, suppose $M \models \bigwedge\!\!\!\bigwedge p(a,b)$. Then $\lceil a,b\rfloor \in \mathcal{B}$. Since $\mathscr{Z}_{\mathcal{B}}$ satisfies part (ii) in the Proposition, we are done.

Therefore, $\mathscr{Z}_{\mathcal{B}}$ is really a species.

It remains to prove that $\mathscr{Z}_{\mathcal{B}}$ is the biggest species satisfying (i) and (ii). Let $\mathcal{Z}$ be a species satisfying (i) and (ii) in the Proposition. Let $a,b \in M^*$ and $I \in \mathcal{Z}$ such that $a \in I < b$. Then by property (i) for $\mathcal{Z}$, we see that $\lceil a,b\rfloor \in \mathcal{B}$. Since the choice of $a,b \in M^*$ and $I \in \mathcal{Z}$ were arbitrary, we get $\mathcal{Z} \subseteq \mathscr{Z}_{\mathcal{B}}$, completing the proof. □

Next, we prove that $\mathscr{B}$ is well-defined.

Proposition 3.7. If $\mathcal{Z}$ is a species, then $\mathscr{B}_{\mathcal{Z}}$ is a notion of intervals.

Proof. Let $\mathcal{Z}$ be a species. By axiom (0) for a species, $\mathcal{Z}$ is nonempty and so

$$\exists I \in \mathcal{Z}\ \exists a, b \in M^*\ (a \in I < b).$$

By the definition of $\mathscr{B}_{\mathcal{Z}}$, we see that $\mathscr{B}_{\mathcal{Z}}$ must then be nonempty.

To prove axiom (1) for notions of intervals for $\mathscr{B}_{\mathcal{Z}}$, let $\lceil a, b \rfloor \in \mathscr{B}_{\mathcal{Z}}$ and $g \in \mathrm{Aut}(M)$ be arbitrary. Using the definition of $\mathscr{B}_{\mathcal{Z}}$, let $I \in \mathcal{Z}$ such that $a \in I < b$. By axiom (1) for species, $I^g \in \mathcal{Z}$, but then $I^g \in \lceil a, b \rfloor^g$, and so $\lceil a, b \rfloor^g \in \mathscr{B}_{\mathcal{Z}}$ by the definition of $\mathscr{B}_{\mathcal{Z}}$.

Axiom (2) for $\mathscr{B}_{\mathcal{Z}}$ follows directly from axiom (2) for species for $\mathcal{Z}$. Axiom (3) for $\mathscr{B}_{\mathcal{Z}}$ is obvious from the definition.

It remains to prove axiom (4) for a notion of intervals for $\mathscr{B}_{\mathcal{Z}}$. Let $B \in M$ be arbitrary, and $Y\colon M_{<B} \times M_{<B} \to M$ be an indicator for $\mathcal{Z}$ below B, which exists by Proposition 2.8. Then for all $a, b < B$, we have

$$\begin{aligned} \lceil a, b \rfloor \in \mathscr{B}_{\mathcal{Z}} &\text{ iff } \exists I \in \mathcal{Z}\ (a \in I < b) && \text{by the definition of } \mathscr{B}_{\mathcal{Z}}, \\ &\text{ iff } Y(a, b) > \mathbb{N} && \text{by the choice of } Y. \end{aligned}$$

Therefore, Y is also an indicator for $\mathscr{B}_{\mathcal{Z}}$ below B. By Proposition 2.1, axiom (4) for a notion of intervals is proved. □

Moreover, $\mathscr{B}$ is a left inverse of $\mathscr{Z}$.

Theorem 3.8. If $\mathcal{B}$ is a notion of intervals, then $\mathscr{B}_{(\mathscr{Z}_{\mathcal{B}})} = \mathcal{B}$.

Proof. Take a notion of intervals $\mathcal{B}$. By Proposition 3.6, the collection $\mathscr{Z}_{\mathcal{B}}$ is a species. So it makes sense to talk about $\mathscr{B}_{(\mathscr{Z}_{\mathcal{B}})}$.

Let $\lceil a, b \rfloor \in \mathscr{B}_{(\mathscr{Z}_{\mathcal{B}})}$. By the definition of $\mathscr{B}_{(\mathscr{Z}_{\mathcal{B}})}$,

$$\exists I \in \mathscr{Z}_{\mathcal{B}}\ (a \in I < b).$$

So by Proposition 3.6(i), $\lceil a, b \rfloor \in \mathcal{B}$.

Conversely, suppose $\lceil a, b \rfloor \in \mathcal{B}$. Then $M_{\mathcal{B}}(a) \in \mathscr{Z}_{\mathcal{B}}$ by Proposition 3.1, and $M_{\mathcal{B}}(a) \in \lceil a, b \rfloor$ by the definition of $M_{\mathcal{B}}(a)$. So

$$\exists I \in \mathscr{Z}_{\mathcal{B}} \ (a \in I < b).$$

Therefore $\lceil a, b \rfloor \in \mathscr{B}_{(\mathscr{Z}_{\mathcal{B}})}$ by the definition of $\mathscr{B}_{(\mathscr{Z}_{\mathcal{B}})}$. □

However, $\mathscr{Z}$ may not be a right inverse of $\mathscr{B}$. This is because of the existence of *symbiotic* species.

Definition. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be species. We say that $\mathcal{Z}_1$ is *symbiotic* with $\mathcal{Z}_2$ if and only if $\mathscr{B}_{\mathcal{Z}_1} = \mathscr{B}_{\mathcal{Z}_2}$.

Fact 3.9. Symbiosis is an equivalence relation on the collection of all species. □

Note that our definition of symbiosis coincides with that first given by Kirby and Paris (1977). Many examples of symbiosis are known. For example, semiregular cuts are symbiotic with regular cuts; strong cuts are symbiotic with cuts satisfying PA; etc. Here we give some examples where the two symbiotic species are disjoint from each other.

Example 3.10. Let $\mathcal{B}$ be a notion of intervals. Set

$$\mathcal{Z}^{(\mathcal{B})} = \{M_{\mathcal{B}}(a) \in \mathscr{Z}_{\mathcal{B}} : M_{\mathcal{B}}(a) \text{ exists and } a \in M^*\}, \text{ and}$$
$$\mathcal{Z}^{[\mathcal{B}]} = \{M_{\mathcal{B}}[b] \in \mathscr{Z}_{\mathcal{B}} : M_{\mathcal{B}}[b] \text{ exists and } b \in M^*\}.$$

Proposition 3.1 and some simple verifications show that that $\mathcal{Z}^{(\mathcal{B})}$ and $\mathcal{Z}^{[\mathcal{B}]}$ are species that are symbiotic with each other. By Lemma 3.5, $\mathcal{Z}^{(\mathcal{B})}$ and $\mathcal{Z}^{[\mathcal{B}]}$ are disjoint. It is easy to see that if M is countable, then $\mathcal{Z}^{(\mathcal{B})}$ and $\mathcal{Z}^{[\mathcal{B}]}$ have order types $1 + \mathbb{Q}$ and $\mathbb{Q} + 1$ respectively. This fact, in the context of elementary cuts, was first noted by Kotlarski (1983, Corollary 6).

Example 3.11 (Kotlarski (1983)). Suppose M is countable and recursively saturated. It can be proved that the sets

$$\mathcal{Z}_1 = \{I \in \mathcal{Z}^{\text{elem}} : I \text{ is recursively saturated}\} \text{ and}$$
$$\mathcal{Z}_2 = \{I \in \mathcal{Z}^{\text{elem}} : I \text{ is not recursively saturated}\}$$

are species that are symbiotic with each other. It follows from the previous example that $\mathcal{Z}_2$ has order type $1+\mathbb{Q}$ (by Theorem 8.4). It is also known that $\mathcal{Z}_1$ is of order type $\mathbb{R}+1$.

As usual, we factor out the collection of all species by the symbiosis relation to get an exact correspondence. The nice thing in our particular case is that there is a natural representative in each equivalence class.

Definition. A class $\mathcal{Z} \subseteq \mathcal{C}$ is a *complete species* if and only if $\mathcal{Z} = \mathscr{Z}_{\mathcal{B}}$ for some notion of intervals $\mathcal{B}$. If $\mathcal{Z}'$ is a species, then the *completion of* $\mathcal{Z}'$, denoted by $\hat{\mathcal{Z}}'$, is the complete species $\mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}'})}$.

Lemma 3.12. Let $\mathcal{Z}$ be a species. Then $\hat{\mathcal{Z}}$ is the unique complete species containing $\mathcal{Z}$ that is symbiotic with $\mathcal{Z}$.

Proof. Let $\mathcal{Z}$ be a species. Then $\hat{\mathcal{Z}} = \mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}})}$ is clearly a complete species.

Let $I \in \mathcal{Z}$ be arbitrary. Then we know that

$$\overset{*}{\forall} a \in I \; \overset{*}{\forall} b > I \; (\lceil a, b \rfloor \in \mathscr{B}_{\mathcal{Z}})$$

by the definition of $\mathscr{B}_{\mathcal{Z}}$. So $I \in \mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}})}$ by the definition of $\mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}})}$. Hence $\mathcal{Z} \subseteq \mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}})} = \hat{\mathcal{Z}}$.

Also $\mathscr{B}_{\hat{\mathcal{Z}}} = \mathscr{B}_{(\mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}})})} = \mathscr{B}_{\mathcal{Z}}$ by Theorem 3.8. So $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ are symbiotic.

Let $\mathcal{Z}'$ be a complete species containing $\mathcal{Z}$ that is symbiotic with $\mathcal{Z}$. Using completeness, let $\mathcal{B}'$ be a notion of intervals such that $\mathcal{Z}' = \mathscr{Z}_{\mathcal{B}'}$. Now

$$\begin{aligned} \mathcal{B}' &= \mathscr{B}_{(\mathscr{Z}_{\mathcal{B}'})} && \text{by Theorem 3.8,} \\ &= \mathscr{B}_{\mathcal{Z}'} && \text{by the choice of } \mathcal{B}', \\ &= \mathscr{B}_{\mathcal{Z}} && \text{since } \mathcal{Z}' \text{ is symbiotic with } \mathcal{Z}. \end{aligned}$$

So, $\mathcal{Z}' = \mathscr{Z}_{\mathcal{B}'} = \mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}})} = \hat{\mathcal{Z}}$, proving uniqueness. □

Finally, we get the promised correspondence.

Theorem 3.13. The maps

$$\mathcal{Z} \mapsto \mathscr{B}_{\mathcal{Z}}$$
$$\mathscr{Z}_{\mathcal{B}} \leftarrow\!\shortmid \mathcal{B}$$

are bijections between the collection of all notions of intervals and the collection of all complete species, and they are inverse to each other.

Proof. The only thing to prove is that if $\mathcal{Z}$ is a complete species, then $\mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}})} = \mathcal{Z}$. However, this is a trivial consequence of Lemma 3.12, because if $\mathcal{Z}$ is a complete species, then $\mathcal{Z}$ is definitely a complete species containing $\mathcal{Z}$ that is symbiotic with $\mathcal{Z}$, and so by uniqueness, $\mathcal{Z} = \hat{\mathcal{Z}} = \mathscr{Z}_{(\mathscr{B}_{\mathcal{Z}})}$. □

Via Lemma 3.12 and the above theorem, we see that all cuts related a notion of intervals $\mathcal{B}$ are actually elements of $\mathscr{Z}_{\mathcal{B}}$. In later chapters, we will actually consider a complete species as our universe of discourse.

While Theorem 3.13 is a nice correspondence, the relationship between notions of intervals and species is more intimate than this. Recall at the beginning of this chapter, we mentioned that notions of intervals are like habitats where species can live in. The following proposition expresses this in a mathematical way.

Proposition 3.14. Let $\mathcal{Z}$ be a species. Then $\mathscr{B}_{\mathcal{Z}}$, considered as a collection of subsets of $\mathcal{Z}$, is a basis for a topology on $\mathcal{Z}$.

Proof. Let $\mathcal{Z}$ be a species. It suffices to prove the following:

(a) $\bigcup \mathscr{B}_{\mathcal{Z}} = \mathcal{Z}$, and

(b) $\forall [a,b],[u,v] \in \mathscr{B}_{\mathcal{Z}}\ ([a,b]_{\mathcal{Z}} \cap [u,v]_{\mathcal{Z}} = \varnothing \text{ or } [a,b]_{\mathcal{Z}} \cap [u,v]_{\mathcal{Z}} \in \mathscr{B}_{\mathcal{Z}})$.

Note that by axiom (0) for a species and the definition of $\mathscr{B}_{\mathcal{Z}}$, we know that $[0,\infty] \in \mathscr{B}_{\mathcal{Z}}$. So (a) holds. The proof of (b) is a routine verification in a number of cases, and is therefore left to the reader. □

This topology on a species plays a very important role in later chapters. In this chapter, we take this point of view to describe the close relationship between the above notion of completeness and that regarding the real line.

Lemma 3.15. Fix a notion of intervals $\mathcal{B}$.

(a) If $(a_n)_{n\in\mathbb{N}}$ is a sequence in M such that $a_0 \ll a_1 \ll a_2 \ll \cdots$, then

$$\sup\{a_n : n \in \mathbb{N}\} \in \mathscr{Z}_{\mathcal{B}}.$$

(b) If $(b_n)_{n\in\mathbb{N}}$ is a sequence in M such that $b_0 \gg b_1 \gg b_2 \gg \cdots$, then

$$\inf\{b_n : n \in \mathbb{N}\} \in \mathscr{Z}_{\mathcal{B}}.$$

Proof. Fix a notion of intervals $\mathcal{B}$. We only prove (a). The proof of (b) is similar. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in M such that $a_0 \ll a_1 \ll a_2 \ll \cdots$ and $I = \sup\{a_n : n \in \mathbb{N}\}$. Clearly, $I \neq \varnothing$. By Fact 2.6(c), I does not have a maximum element. So $I \in \mathcal{C}$.

Let $a, b \in M^*$ such that $a \in I < b$. Using the fact that $\{a_n : n \in \mathbb{N}\} \subseteq_{\mathrm{cf}} I$, pick $n \in \mathbb{N}$ such that $a \leqslant a_n$. Now

$$a \leqslant a_n \ll a_{n+1} \in I < b.$$

Hence by Fact 2.6(d), $a \ll b$, i.e., $\lceil a, b \rfloor \in \mathcal{B}$. Therefore, $I \in \mathscr{Z}_{\mathcal{B}}$. $\square$

Corollary 3.16. Let $\mathcal{B}$ be a notion of intervals.

(a) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in M^*. If

- $a_0 \not\gg a_1 \not\gg a_2 \not\gg \cdots$ and
- $(a_n)_{n\in\mathbb{N}}$ contains a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ such that $a_{n_0} \ll a_{n_1} \ll a_{n_2} \ll \cdots$,

then $\sup\{a_n : n \in \mathbb{N}\} \in \mathscr{Z}_{\mathcal{B}}$.

(b) Let $(b_n)_{n\in\mathbb{N}}$ be a sequence in M^*. If

- $b_0 \not\ll b_1 \not\ll b_2 \not\ll \cdots$ and
- $(b_n)_{n\in\mathbb{N}}$ contains a subsequence $(b_{n_i})_{i\in\mathbb{N}}$ such that $b_{n_0} \gg b_{n_1} \gg b_{n_2} \gg \cdots$,

then $\inf\{b_n : n \in \mathbb{N}\} \in \mathscr{Z}_{\mathcal{B}}$. $\square$

Another face of species completeness will be described in the next chapter.

CHAPTER 4

ENFORCEABLE PROPERTIES

> SECOND ACT OF INTUITIONISM *Admitting two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired* [...]; *secondly in the shape of mathematical species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be 'equal' to it, definitions of equality having to satisfy the conditions of symmetry, reflexivity and transitivity.*
>
> Luitzen Egbertus Jan Brouwer (1981)
> **Brouwer's Cambridge Lectures on Intuitionism**, Chapter 1, p. 8
> A series of lectures from 1946 to 1951

The idea of equipping a family of cuts with a topology originated historically from the work of Henryk Kotlarski (1984b, see also the appendix in Smoryński (1982b)) in the 1980s. In GCMA, Kaye studies topological properties of sets of cuts within a similar framework. We follow Kaye's development in this chapter, and show that everything that works there works in the present setting as well.

Throughout this chapter, we assume countability of our model M. We work in a fixed notion of intervals $\mathcal{B}$, and its corresponding complete species $\mathcal{Z} = \mathscr{X}_{\mathcal{B}}$.

Theorem 4.1. $\mathcal{Z}$ is homeomorphic to the Cantor space 2^ω.

Proof. We use a tree argument that is long and routine. However, the idea is simple. We construct a tree of nested intervals in which each infinite path determines a unique cut. With some care in the construction, this defines a homeomorphism $2^\omega \to \mathcal{Z}$.

Fix an enumeration $(x_n)_{n\in\mathbb{N}}$ of M.

Define the sequence $([a_\sigma, b_\sigma])_{\sigma\in 2^{<\omega}}$ of $\mathcal{B}$-intervals recursively as follows.

- Set $[a_\varnothing, b_\varnothing] = [0, \infty]$. This is an interval because $\mathcal{B}$ satisfies axioms (0) and (3) for a notion of intervals.

- Let $n \in \mathbb{N}$ and $\sigma \in 2^n$ such that $[a_\sigma, b_\sigma]$ is defined. Using axiom (2) for notions of intervals, pick $c_\sigma \in M$ such that $a_\sigma \ll c_\sigma \ll b_\sigma$. Define

$$[a_{\sigma 0}, b_{\sigma 0}] = \begin{cases} [a_\sigma, x_n], & \text{if } a_\sigma \ll x_n \ll b_\sigma; \\ [x_n, c_\sigma], & \text{if } [x_n, c_\sigma]_{\mathcal{Z}} = [a_\sigma, c_\sigma]_{\mathcal{Z}}; \\ [a_\sigma, c_\sigma], & \text{otherwise;} \end{cases}$$

  and

$$[a_{\sigma 1}, b_{\sigma 1}] = \begin{cases} [x_n, b_\sigma], & \text{if } a_\sigma \ll x_n \ll b_\sigma; \\ [c_\sigma, x_n], & \text{if } [c_\sigma, x_n]_{\mathcal{Z}} = [c_\sigma, b_\sigma]_{\mathcal{Z}}; \\ [c_\sigma, b_\sigma], & \text{otherwise.} \end{cases}$$

What remains involves the proofs of the following statements.

(a) $\forall \varepsilon \in 2^\omega \ \sup\{a_{\varepsilon\restriction_n} : n \in \mathbb{N}\} = \inf\{b_{\varepsilon\restriction_n} : n \in \mathbb{N}\}$.

(b) For each $\varepsilon \in 2^\omega$, either $(a_{\varepsilon\restriction_n})_{n\in\mathbb{N}}$ satisfies the hypotheses in Corollary 3.16(a), or $(b_{\varepsilon\restriction_n})_{n\in\mathbb{N}}$ satisfies the hypotheses in Corollary 3.16(b).

(c) The function $f\colon 2^\omega \to \mathcal{Z}$ defined by

$$\forall \varepsilon \in 2^\omega \ f(\varepsilon) = \sup\{a_{\varepsilon\restriction_n} : n \in \mathbb{N}\}$$

  is a homeomorphism.

Clearly, proving these claims suffices. We do this step by step.

(a) Let $\varepsilon \in 2^\omega$. It is immediate from our construction that

$$\sup\{a_{\varepsilon\restriction_n} : n \in \mathbb{N}\} \subseteq \inf\{b_{\varepsilon\restriction_n} : n \in \mathbb{N}\}.$$

Suppose the converse does not hold. Take $m \in \mathbb{N}$ such that $x_m \in \inf\{b_{\varepsilon\restriction_n} : n \in \mathbb{N}\} \setminus \sup\{a_{\varepsilon\restriction_n} : n \in \mathbb{N}\}$. Then in particular,

$$a_{\varepsilon\restriction_m} \leqslant x_m \leqslant b_{\varepsilon\restriction_m}.$$

If $a_{\varepsilon\restriction_m} \ll x_m \ll b_{\varepsilon\restriction_m}$, then $b_{\varepsilon\restriction_m 0} = x_m = a_{\varepsilon\restriction_m 1}$ by construction, and thus $\varepsilon(m+1)$ can neither be 0 nor 1. Hence, we must either have $a_{\varepsilon\restriction_m} \not\ll x_m$ or $x_m \not\ll b_{\varepsilon\restriction_m}$. So

$$[x_m, c_{\varepsilon\restriction_m}]_{\mathcal{Z}} = [a_{\varepsilon\restriction_m}, c_{\varepsilon\restriction_m}]_{\mathcal{Z}} \text{ or } [c_{\varepsilon\restriction_m}, x_m]_{\mathcal{Z}} = [c_{\varepsilon\restriction_m}, b_{\varepsilon\restriction_m}]_{\mathcal{Z}},$$

which in turn implies that

$$x_m = a_{\varepsilon\restriction_m 0} \leqslant a_{\varepsilon\restriction_{m+1}} \text{ or } b_{\varepsilon\restriction_{m+1}} \leqslant b_{\varepsilon\restriction_m 1} = x_m$$

by the construction of our tree of intervals. This contradicts the choice of x_m.

(b) Fix $\varepsilon \in 2^\omega$. By construction,

$$a_{\varepsilon\restriction_0} \not\gg a_{\varepsilon\restriction_1} \not\gg a_{\varepsilon\restriction_2} \not\gg \cdots \text{ and } b_{\varepsilon\restriction_0} \not\ll b_{\varepsilon\restriction_1} \not\ll b_{\varepsilon\restriction_2} \not\ll \cdots.$$

Suppose $(b_{\varepsilon\restriction_n})_{n\in\mathbb{N}}$ does not satisfy the hypotheses in Corollary 3.16(b), and let $m \in \mathbb{N}$ such that

$$b_{\varepsilon\restriction_m} \not\gg b_{\varepsilon\restriction_{m+1}} \not\gg b_{\varepsilon\restriction_{m+2}} \not\gg \cdots.$$

Then we must have $a_{\varepsilon\restriction_m} \ll a_{\varepsilon\restriction_{m+1}} \ll a_{\varepsilon\restriction_{m+2}} \ll \cdots$ because

$$[a_{\varepsilon\restriction_m}, b_{\varepsilon\restriction_m}]_{\mathcal{Z}} \subsetneq [a_{\varepsilon\restriction_{m+1}}, b_{\varepsilon\restriction_{m+1}}]_{\mathcal{Z}} \subsetneq [a_{\varepsilon\restriction_{m+2}}, b_{\varepsilon\restriction_{m+2}}]_{\mathcal{Z}} \subsetneq \cdots$$

by the choice of our tree, as required.

(c) By Corollary 3.16 and the previous steps, the function f is well-defined. Given $I \in \mathcal{Z}$, we let $g(I)$ be the sequence $\varepsilon \in 2^\omega$ defined recursively by: if $n \in \mathbb{N}$ such that $\varepsilon(m)$ is defined for all $m < n$, then

$$\varepsilon(n) = \begin{cases} 0, & \text{if } I \in [a_{\varepsilon\restriction_n 0}, b_{\varepsilon\restriction_n 0}]; \\ 1, & \text{if } I \in [a_{\varepsilon\restriction_n 1}, b_{\varepsilon\restriction_n 1}]. \end{cases}$$

This defines a map $g\colon \mathcal{Z} \to 2^\omega$. It is easy to see that f and g are inverse to each other, and so f is bijective.

For a finite sequence $\sigma \in 2^{<\omega}$, write U_σ for the basic open set in 2^ω that it determines, namely, the set

$$U_\sigma = \{\varepsilon \in 2^\omega : \sigma \text{ is an initial part of } \varepsilon\}.$$

It is straightforward to check that

$$f(U_\sigma) = [a_\sigma, b_\sigma]_{\mathcal{Z}}$$

for each $\sigma \in 2^{<\omega}$, and

$$g([a,b]_{\mathcal{Z}}) = \bigcup\{U_\sigma : \sigma \in 2^{<\omega} \text{ with } [a_\sigma, b_\sigma] \subseteq [a,b]\}$$

for each $[a,b] \in \mathcal{B}$. Therefore, f and g are both continuous. □

The previous theorem makes a whole range of topological tools available to us. For example, we now know that $\mathcal{Z}$, as a topological space, is perfect, compact, totally disconnected, of cardinality $2^{\aleph_0}$, and homeomorphic to a complete metric space. In addition, the Baire Category Theorem applies.

Definition. Let X be a topological space. A subset $A \subseteq X$ is *corare in* X if and only if the intersection of A with any nonempty open set in X has a nonempty interior. The subset A is *comeagre in X* if and only if it contains a countable intersection of corare sets.

Baire Category Theorem. In a complete metric space, every comeagre subset is dense.

Proof. Please see Theorem 9.1 in Oxtoby (1971), say. □

Recall that dense subsets of a complete species are exactly those that are *indicated* in the sense of Kirby and Paris (1977). This is what makes us interested in comeagre sets of cuts. However, it should be noted that there are clearly dense sets of cuts that are not comeagre. In spite of this, we still study comeagre sets because they have many nice properties, including a useful game-theoretic characterization. (See also Corollary 6.3.)

Definition. The *Banach–Mazur game on $\mathcal{B}$* is the following game.

- There are two players, called $\forall$ and $\exists$.
- Starting from $\forall$, the two players alternatingly choose a $\mathcal{B}$-interval that is a subinterval of previously chosen ones.
- The game terminates in ω many steps.

A play of this game gives rise to a sequence $([a_n, b_n])_{n\in\mathbb{N}}$. The cut $\sup\{a_n : n \in \mathbb{N}\}$ is called the *outcome* of the play, and we always assume this cut to lie in $\mathscr{Z}$.

A property P of cuts is *enforceable* if and only if $\exists$ has a way to ensure the outcome of a play has property P. Similarly, a subset $\mathcal{P}$ of $\mathscr{Z}$ is *enforceable* if and only if the property of being an element of $\mathcal{P}$ is enforceable.

Theorem 4.2 (Banach). A set $\mathcal{P} \subseteq \mathscr{Z}$ is enforceable if and only if it is comeagre in $\mathscr{Z}$.

Proof. Please see Theorem 6.1 in Oxtoby (1971), say. □

The following standard results about comeagre sets are also useful. Their proofs can again be found in Oxtoby (1971).

Fact 4.3. (a) Every enforceable class of cuts is of size continuum.

(b) A countable conjunction of enforceable properties is enforceable.

(c) Every co-countable subset in $\mathcal{Z}$ is enforceable. □

Kaye proves in GCMA the enforceability of numerous properties about cuts. Some of them can be generalized to the current setting. The two examples below are easy consequences of Fact 4.3(c).

Proposition 4.4. It is enforceable that a $\mathcal{Z}$-cut is not an ω-limit. □

Proposition 4.5. It is enforceable that

$$I \neq M_{\mathcal{B}}(a) \text{ and } I \neq M_{\mathcal{B}}[a] \text{ whenever } a \in M$$

for a $\mathcal{Z}$-cut I. □

Other proofs involve Banach–Mazur games. For example, one can enforce some non-saturation of the expansion of M by the outcome of a play. We present the best counterexample we can find here.

Definition. Let $\mathscr{L}_{\mathrm{A}}^{*}$ denote the language obtained from $\mathscr{L}_{\mathrm{A}}$ by adding an extra unary relation symbol. We do not specify what this new symbol is as long as no ambiguity arises, and use the name of the set it interprets to refer to it. The language obtained from $\mathscr{L}_{\mathrm{A}}^{*}$ by adding all $\mathscr{L}_{\mathrm{A}}$ Skolem functions is denoted by $\mathscr{L}_{\mathrm{Sk}}^{*}$. We write $\forall\exists(\mathscr{L}_{\mathrm{Sk}}^{*})$ for the set of all $\forall\exists$ formulas in the language $\mathscr{L}_{\mathrm{Sk}}^{*}$.

Proposition 4.6. It is enforceable that $\mathbb{N}$ is $\forall\exists(\mathscr{L}_{\mathrm{Sk}}^{*})$ definable with parameters in (M, I) for a $\mathcal{Z}$-cut I.

Proof. We play a Banach–Mazur game on $\mathcal{B}$. Suppose $\forall$ plays $[a, b]$ in his first move. Without loss of generality, assume $b < \infty$. Using Proposition 2.1, let $Y \in M$ be a monotone indicator for $\mathcal{B}$ below $b+1$. We show that $\exists$ can make the outcome of the play I satisfy

$$\{n \in M : M \models \forall x \in I\ \exists y \in I\ Y(x, y) \geqslant n\} = \mathbb{N}.$$

Note that since $I \in \mathscr{Z}_{\mathcal{B}}$, it is clear that $\{n \in M : M \models \forall x \in I\ \exists y \in I\ Y(x, y) \geqslant n\} \supseteq \mathbb{N}$ for each outcome I. Let $n \in M$ be nonstandard, and suppose that $\exists$ is given $[u, v] \subseteq [a, b]$ to play in. Using Lemma 2.7, let $\exists$ play $[x_n, y_n] \subseteq [u, v]$ such that $Y(x_n, y_n) < n$. Using the countability of M, player $\exists$ can do this for every nonstandard $n \in M$ in a single play. Now, if I is an outcome of this play and $n \in M$ is nonstandard, then we have $x_n \in I < y_n$ and so

$$Y(x_n, y) \leqslant Y(x_n, y_n) < n$$

for each $y \in I$ by the monotonicity of Y. This proves our claim. □

Remark. In Kirby's (1977, Definition 4.5) words, the above proof shows that one can enforce the *index* of a cut corresponding to an indicator to be $\mathbb{N}$.

Corollary 4.7. It is enforceable that (M, I) is not $\forall\exists(\mathscr{L}^*_{\mathrm{Sk}})$ recursively saturated for a $\mathcal{Z}$-cut I.

Proof. Let I be a cut. Suppose $\mathbb{N}$ is $\forall\exists(\mathscr{L}^*_{\mathrm{Sk}})$ definable with parameters in (M, I). We show that (M, I) cannot be $\forall\exists(\mathscr{L}^*_{\mathrm{Sk}})$ recursively saturated, which suffices by Proposition 4.6.

Take an element $Y \in M$ and an $\forall\exists(\mathscr{L}^*_{\mathrm{Sk}})$ formula $\Phi(x, y, X)$ with a second order variable X such that

$$M \models \Phi(x, Y, I) \text{ iff } x \in \mathbb{N}$$

for every $x \in M$. Then the recursive $\forall\exists(\mathscr{L}^*_{\mathrm{Sk}})$ type

$$\{\Phi(x, Y, I) \wedge x > n : n \in \mathbb{N}\}$$

is finitely satisfied but not realized in (M, I). □

Question 4.8. How much saturation can we enforce in the structure (M, I) for a $\mathcal{Z}$-cut I? Can $\exists\forall(\mathscr{L}^*_{\mathrm{Sk}})$ recursive saturation be enforced? Note that there is a similar result in GCMA.

Enforceability results related to the Kirby–Paris notions of semiregularity and regularity are proved in GCMA. The proofs adapt easily to the present context.

Definition. A notion of intervals $\mathcal{B}$ is said to be *relatively indestructible* if and only if for every finite $[a,b] \in \mathcal{B}$, there is an element $c \in M$ such that

$$a = (c)_0 \ll (c)_1 \ll \cdots \ll (c)_{a+1} = b.$$

Just like the notion of *relative largeness* in the Paris–Harrington statement, relative indestructibility does not fit into the rest of the picture at all, at least in the sense of Friedman (1999). This is where the word "relative" comes from. Anyway, this notion does work, which may serve to justify its appearance.

Proposition 4.9. Semiregularity is enforceable if and only if $\mathcal{B}$ is relatively indestructible.

Proof. Suppose $\mathcal{B}$ is not relatively indestructible. Pick a witness $[a,b] \in \mathcal{B}$ to this. We need a monotone indicator for $\mathcal{B}$ below $b+1$. In fact, we want one with stronger monotonicity properties that is in a slightly different form, namely, we want a function

$$\begin{aligned} F\colon \quad & M^2 \to M \\ & (n,x) \mapsto F_n(x) \end{aligned}$$

definable in M with parameters such that

- for all $x, y < b+1$, we have $x \ll y$ if and only if there exists a nonstandard $n \in M$ such that $F_n(x) < y$;
- for every $x \in M$, the function $n \mapsto F_n(x)$ from M to M has cofinal image in M;
- F is strictly increasing in its first argument;
- F is non-decreasing in its second argument; and
- F always increases its second argument.

These requirements give us reasonable control over the situation even when we go out of bound. The existence of such a function can be found using Proposition 2.1. Since a similar construction appears in GCMA as well, we omit this part of the proof and simply fix one such F.

If $n \in M$ is nonstandard such that $F_n^{(a+1)}(a) \leqslant b$, then the coded sequence

$$a = F_n^{(0)}(a) \ll F_n^{(1)}(a) \ll \cdots \ll F_n^{(a+1)}(a) \leqslant b$$

splits the $[a, b]$ into $a+1$ subintervals, contradicting the choice of $[a, b]$. So for all nonstandard $n \in M$, we have $F_n^{(a+1)}(a) > b$. Using underspill, pick $k \in \mathbb{N}$ such that $F_k^{(a+1)}(a) > b$.

We claim that no $\mathcal{Z}$-cut in $[a, b]$ is semiregular. This proves the proposition because if $\forall$ plays $[a, b]$ in his first move, then there is no way $\exists$ can make the outcome semiregular.

Let $I \in [a, b]$ be an arbitrary $\mathcal{Z}$-cut. Note that since $\mathcal{Z} = \mathscr{X}_{\mathcal{B}}$ and k is standard, I must be closed under F_k. Consider the coded function $f\colon M_{\leqslant a+1} \to M$ defined by

$$f(x) = F_k^{(x)}(a)$$

for each $x \leqslant a + 1$. By our choice of F, the function f is strictly increasing. If $B \in I$ is an upper bound for $\operatorname{Im}(f) \cap I$, then by induction, the set $\operatorname{Im}(f) \cap I = \operatorname{Im}(f) \cap M_{\leqslant B}$ has a maximum element, which is a not possible because I is closed under F_k and $F_k^{(a+1)}(a) > b$. Hence, $\operatorname{Im}(f) \cap I \subseteq_{\mathrm{cf}} I$, as required.

Conversely, suppose $\mathcal{B}$ is relatively indestructible. We devise a strategy for $\exists$ in the Banach–Mazur game to make the outcome semiregular. Using the countability of M, we arrange each coded function to be dealt with in infinitely many steps.

Suppose $\exists$ is given the interval $[a, b]$ to play in, and we are considering the coded function $f\colon M_{<d} \to M$ in the current step where $d \in M$. If $d > a$, then there is nothing $\exists$ has to do, and she can just play $[a, b]$. Suppose $d \leqslant a$. By relative indestructibility of $\mathcal{B}$, let $c \in M$ such that

$$a = (c)_0 \ll (c)_1 \ll \cdots \ll (c)_{a+1} = b.$$

By the pigeonhole principle, we see that there is an $i \leqslant a$ such that $[(c)_i, (c)_{i+1}]$ is disjoint with $\mathrm{Im}(f)$. Player $\exists$ plays this interval in this move. It is then clear that no matter what the outcome I is, $\mathrm{Im}(f) \cap I$ cannot be cofinal in I. $\square$

Proposition 4.10. The property of being not regular is enforceable.

Proof. This involves a delicate combinatorial argument that is essentially the same as that in Theorem 4.15 in GCMA. So I do not reproduce it here. The proof of the previous proposition should give enough hints on how the proof in GCMA can be modified to suit to current situation. $\square$

CHAPTER 5

PREGENERICITY AND GENERICITY

> This paper is primarily concerned with a special case of one of the leading problems of mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula. But in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest and are most conveniently set out by themselves beforehand.
>
> Frank P. Ramsey (1930)
> On a problem of formal logic

Let $(\theta_i(x, y, z))_{i\in\mathbb{N}}$ be a recursive enumeration of $\mathscr{L}_{\mathrm{A}}$ formulas in the free variables x, y, z. Fix a notion of intervals $\mathcal{B}$ and its corresponding species $\mathcal{Z} = \mathscr{Z}_{\mathcal{B}}$.

In GCMA, Kaye shows the existence of intervals with a self-similarity property. He calls such intervals *constant intervals.*

Definition. Let $c \in M$. A finite $\mathcal{B}$-interval $[a, b]$ is *constant over c (with respect to $\mathcal{B}$)* if and only if

$$\forall x \in [a, b]\ \forall [u, v] \subseteq [a, b]\ \exists x' \in [u, v]\ \ \mathrm{tp}(x, c) = \mathrm{tp}(x', c).$$

On the face of it, constant intervals are not sufficiently self-similar to carry out the back-and-forth argument needed there. So Kaye goes on to define an additional assumption on intervals.

Definition. Let $[a, b]$ be a finite $\mathcal{B}$-interval, Y be an indicator for $\mathcal{B}$ below $b + 1$, and $c \in M$. Then $[a, b]$ is *small over c (for Y)* if and only if $Y(a, b) < x$ for each nonstandard $x \in \mathrm{cl}(a, c)$.

Smallness is not a robust notion in the current context because it is sensitive to the choice of the indicator. Moreover, its name is slightly misleading.

Proposition 5.1. Let $b, c \in M$ and Y be a monotone indicator for $\mathcal{B}$ below $b+1$. Then every $\mathcal{B}$-interval in $M_{\leqslant b}$ contains a subinterval that is not small over c for Y. In particular, every small interval over c in $M_{\leqslant b}$ contains a subinterval that is not small over c.

Proof. Let $c \in M$ and $[a, b] \in \mathcal{B}$ be finite. Fix a monotone indicator Y for $\mathcal{B}$ below $b+1$. Without loss of generality, assume a is nonstandard, or else make $[a, b]$ smaller using axiom (2) for intervals and Fact 2.6(b). Using axiom (2) again, let $d \in [a, b]$ such that $a \ll d \ll b$. Choose a nonstandard $m \in M$ such that

$$m < \min\big(\lfloor \log_2(a) \rfloor, \lfloor \log_2(d-a) \rfloor, Y(d, b)\big),$$

which is possible since $Y(d, b) > \mathbb{N}$.

Note that $2^m < 2^{\lfloor \log_2(a) \rfloor} \leqslant a$. So there is an odd $w \in M$ such that $2^m w < a$. Let

$$x = (\max w)\big(\exists y(w = 2y+1) \wedge 2^m w < a\big).$$

Then $2^m(x+2) \geqslant a$ by the maximality of x. Also,

$$2^m(x+2) = 2^m x + 2^{m+1} < a + 2^{\lfloor \log_2(d-a) \rfloor} \leqslant a + d - a = d.$$

Hence $2^m(x+2) \in [a, d]$. Let $a' = 2^m(x+2)$. Note that $a' \leqslant d \ll b$ and so by Fact 2.6(d), $[a', b]$ is a $\mathcal{B}$-interval. However, by the monotonicity of Y,

$$Y(a', b) \geqslant Y(d, b) > m = (\max w)(2^w \mid a') \in \mathrm{cl}(a', c) \setminus \mathbb{N}.$$

Therefore, $[a', b]$ is not small over c. □

We solve this problem by introducing a two-variable version of constant intervals. We call intervals having this stronger self-similarity property *pregeneric intervals*.

Definition. For $x, y, x', y', c \in M$, we write $(x, y, c) \equiv (x', y', c)$ to mean

$$\forall i \in \mathbb{N}\ (\theta_i(x, y, c) \leftrightarrow \theta_i(x', y', c)).$$

Definition. Let $c \in M$. A finite $\mathcal{B}$-interval $[a, b]$ is *pregeneric over* c *(with respect to* $\mathcal{B}$*)* if and only if

$$\forall x, y \in [a, b]\ \forall [u, v] \subseteq [a, b]\ \exists x', y' \in [u, v]\ (x, y, c) \equiv (x', y', c).$$

We say that a $\mathcal{B}$-interval is *pregeneric (with respect to* $\mathcal{B}$*)* if and only if it is pregeneric over 0.

Remark. Pregenericity implies smallness in the context of GCMA indicators.

Similar to constant intervals, plenty of pregeneric intervals exist in countable arithmetically saturated models of PA. The proof of this is essentially the same, but instead of presenting this proof straightaway, we study it step-by-step in finer details. Such closer investigation reveals that although arithmetic saturation is essential on the whole, a large part of the proof goes through without any countability or saturation assumption.

Our main approach is *finitization*. The proofs are technical in the sense that they are flooded with parameters. Otherwise, the main ideas are straightforward.

Definition. For $x, y, x', y', c \in M$ and $n \in \mathbb{N}$, write $(x, y, c) \equiv_n (x', y', c)$ to mean

$$\forall i \leqslant n\ (\theta_i(x, y, c) \leftrightarrow \theta_i(x', y', c)).$$

Remark. Note that "$(x, y, z) \equiv_n (x', y', z)$" is equivalent to a parameter free formula in (M, Sat) for any partial inductive satisfaction class Sat for M.

Here comes the moment when we start using axiom (4) for a notion of intervals.

Definition. Let $\lceil a, b \rfloor$ be a finite semi-interval, $n, k, c \in M$, and Y be an indicator for $\mathcal{B}$ below $b + 1$. We say that $\lceil a, b \rfloor$ is $(n, k)_Y$*-pregeneric over* c *(with respect to* $\mathcal{B}$*)* if and only if $Y(a,b) \geqslant k$ and

$$\forall x, y \in \lceil a, b \rfloor\ \forall \lceil u, v \rfloor \subseteq \lceil a, b \rfloor\ \big(Y(u,v) \geqslant k \to \exists x', y' \in \lceil u, v \rfloor\ ((x,y,c) \equiv_n (x', y', c))\big).$$

We shall omit the subscript Y if the indicator in consideration is clear from the context.

To prove the existence of (n,k)-pregeneric intervals, we use the tree argument given in GCMA. The only difference here is that the tree is now finite.

Technical definition. Let $\lceil a, b \rfloor$ be a finite semi-interval, $c \in M$, and Y be a monotone indicator for $\mathcal{B}$ below $b+1$. For $i \in \mathbb{N}$, define the function $e_i \colon M_{\leqslant b} \times M_{\leqslant b} \to M$ by

$$\forall r, s \leqslant b\ \ e_i(r,s) = \max \big\{ l \in M : \exists \lceil r', s' \rfloor \subseteq \lceil r, s \rfloor \\ \big(Y(r', s') = l \wedge \forall x, y \in \lceil r', s' \rfloor\ \neg\theta_i(x,y,c)\big)\big\}.$$

The *tree of possibilities from $\lceil a, b \rfloor$ over c with respect to Y* is a sequence $(\lceil r_\sigma, s_\sigma \rfloor)_{\sigma \in 2^{<\omega}}$ of semi-intervals defined recursively as follows.

- Set $\lceil r_\varnothing, s_\varnothing \rfloor = \lceil a, b \rfloor$.
- Let $m \in \mathbb{N}$ and $\sigma \in 2^m$ such that $\lceil r_\sigma, s_\sigma \rfloor$ is defined. Set $\lceil r_{\sigma 0}, s_{\sigma 0} \rfloor = \lceil r_\sigma, s_\sigma \rfloor$ and let $\lceil r_{\sigma 1}, s_{\sigma 1} \rfloor \subseteq \lceil r_\sigma, s_\sigma \rfloor$ such that

$$r_{\sigma 1} = (\mu r \in \lceil r_\sigma, s_\sigma \rfloor)\big(\exists s \in \lceil r_\sigma, s_\sigma \rfloor \\ \big(Y(r,s) \geqslant e_m(r_\sigma, s_\sigma) \wedge \forall x, y \in \lceil r, s \rfloor\ \neg\theta_m(x,y,c)\big)\big), \text{ and}$$

$$s_{\sigma 1} = (\max s \in \lceil r_\sigma, s_\sigma \rfloor)\big(\forall x, y \in \lceil r_{\sigma 1}, s \rfloor\ \neg\theta_m(x,y,c)\big).$$

Remark. Note that the function e defined above is dependent only on the choice of $c \in M$ and the indicator Y. Also, both e and the tree of possibilities are uniformly definable in (M, Sat) for any partial inductive satisfaction class Sat for M. This is true for $(n,k)_Y$-pregenericity over an element c of M as well.

The idea of the proof is that given a large enough finite semi-interval $\lceil a, b \rfloor$ and a formula $\theta(x, y)$, exactly one of two things has to happen: either there is a large subinterval of $\lceil a, b \rfloor$ in which no pair of elements satisfy $\theta(x, y)$, or there is not. In the first case, the witnessing subinterval is homogeneous for $\theta(x, y)$, simply because no pair of elements in there satisfies this formula. In the second case, the whole semi-interval is already homogeneous for $\theta(x, y)$, because by assumption, every large enough subinterval contains a pair of elements satisfying $\theta(x, y)$. In either case, we get a sufficiently large subinterval that is homogeneous for $\theta(x, y)$.

We can repeat this argument with all $\mathscr{L}_{\mathrm{A}}$ formulas. It is in general quite hard to find out which case we are in, but we definitely know what possibilities we have. This gives rise to the tree of possibilities defined above. We do not need to know which way down the tree we have to go. We only need to know there is a way that works.

Technical lemma 5.2. Let $\lceil a, b \rfloor$ be a finite semi-interval, Y be a monotone indicator for $\mathcal{B}$ below $b + 1$, and $c, k \in M$ such that $Y(a, b) \geqslant k$. If $(\lceil r_\sigma, s_\sigma \rfloor)_{\sigma \in 2^{<\omega}}$ is the tree of possibilities from $\lceil a, b \rfloor$ over c with respect to Y, then we have

$$\forall m \in \mathbb{N}\ \exists! \sigma \in 2^m\ \big(Y(r_\sigma, s_\sigma) \geqslant k \land \forall i < m\ (\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) < k)\big).$$

Proof. This can be proved by an easy induction on m. □

It is then down to checking how many formulas we need to guarantee a certain amount of pregenericity.

Technical definition. Let $\beta\colon \mathbb{N} \to \mathbb{N}$ be the function defined by: for all $n \in \mathbb{N}$, the number $\beta(n)$ is the least $m \in \mathbb{N}$ such that

> if $\phi(x, y, z)$ is a Boolean combination of formulas in $\{\theta_i(x, y, z) : i \leqslant n\}$, then there is a formula $\phi'(x, y, z) \in \{\theta_i(x, y, z) : i \leqslant m\}$ that is logically equivalent to $\phi(x, y, z)$ in M.

Theorem 5.3. Let n be a natural number, $\lceil a, b\rfloor$ be a finite semi-interval, $k, c \in M$, and Y be a monotone indicator for $\mathcal{B}$ below $b+1$ such that $Y(a, b) \geqslant k$. Then $\lceil a, b\rfloor$ contains a semi-interval that is $(n, k)_Y$-pregeneric over c with respect to Y. Moreover, if Sat is a partial inductive satisfaction class for M, then one such semi-interval is definable in (M, Sat) uniformly in the parameters a, b, c, Y, n, k.

Proof. Let $\lceil a, b\rfloor$ be a finite semi-interval, $k, c \in M$, and Y be a monotone indicator for $\mathcal{B}$ below $b+1$ such that $Y(a, b) \geqslant k$. Let $(\lceil r_\sigma, s_\sigma\rfloor)_{\sigma\in 2^{<\omega}}$ be the tree of possibilities from $\lceil a, b\rfloor$ over c with respect to Y. Using Lemma 5.2, define the function $\pi\colon \mathbb{N} \to 2^{<\omega}$ by setting $\pi(m)$ to be the unique $\sigma \in 2^m$ such that

$$Y(r_\sigma, s_\sigma) \geqslant k \wedge \forall i < m\ (\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) < k)$$

for each $m \in \mathbb{N}$. It can then be checked that $\lceil r_{\pi(\beta(n))}, s_{\pi(\beta(n))}\rfloor \subseteq \lceil a, b\rfloor$ is $(n, k)_Y$-pregeneric over c for every $n \in \mathbb{N}$.

The "moreover" part can be proved by a careful check of all the steps, and is left to the reader. □

By noting that almost everything in the above argument is coded in M, one can prove the same statement about fully pregeneric intervals in a similar way.

Theorem 5.4. Suppose M is arithmetically saturated and $c \in M$. Then every $\mathcal{B}$-interval contains a pregeneric subinterval over c.

Proof. Suppose M is arithmetically saturated. Let $c \in M$ and $[a, b] \in \mathcal{B}$. Without loss of generality, assume $b < \infty$. Fix a monotone indicator Y for $\mathcal{B}$ below $b+1$, and let $(\lceil r_\sigma, s_\sigma\rfloor)_{\sigma\in 2^{<\omega}}$ be the tree of possibilities from $[a, b]$ over c with respect to Y. By recursive saturation, this tree of possibilities and thus $(Y(r_\sigma, s_\sigma))_{\sigma\in 2^{<\omega}}$ are coded in M. Using the strength of $\mathbb{N}$ in M, let $d > \mathbb{N}$ such that

$$\forall \sigma \in 2^{<\omega}\ (Y(r_\sigma, s_\sigma) > d \Leftrightarrow Y(r_\sigma, s_\sigma) > \mathbb{N}).$$

In particular, $Y(a,b) > d$ since $[a,b] \in \mathcal{B}$. By Lemma 5.2, we have

$$\forall m \in \mathbb{N}\ \exists! \sigma \in 2^m\ \big(Y(r_\sigma, s_\sigma) > d \wedge \forall i < m\ (\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) \leqslant d)\big).$$

Using recursive saturation of M, let $n > \mathbb{N}$ and $\sigma \in 2^n$ such that

$$\begin{aligned} Y(r_\sigma, s_\sigma) > d \wedge \forall i < n\ \big(\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) \leqslant d\big) \\ \wedge \forall i < n\ \big(\lceil r_{\sigma\restriction_i}, s_{\sigma\restriction_i} \rfloor \supseteq \lceil r_{\sigma\restriction_{i+1}}, s_{\sigma\restriction_{i+1}} \rfloor\big). \end{aligned}$$

It can then be checked that $[r_\sigma, s_\sigma] \subseteq [a,b]$ is pregeneric over c. $\square$

In many arguments in the model theory of arithmetic, finite tuples can be assumed to have length one because of the existence of a simple definable pairing function. The distinction between constant intervals and pregeneric intervals seems to be an exception to this. However, we are still unable to find an explicit example demonstrating this.

Problem 5.5. Over some $c \in M$, find a constant interval that is not pregeneric.

Note that if the intersection of two constant intervals is an interval, then the union of them is also constant. On the same line of thought, one may ask whether this is also true for pregeneric intervals.

Problem 5.6. Over some $c \in M$, find two pregeneric intervals whose intersection is an interval but whose union is not pregeneric.

On the other hand, one can try to strengthen the definition of pregeneric intervals to one involving tuples of length bigger than two. After a moment of thought, one may realize that this modification does not give us anything much stronger actually, at least when the model is recursively saturated.

Proposition 5.7. Suppose M is recursively saturated, and let $c \in M$. Then a finite interval $[a,b] \in \mathcal{B}$ is pregeneric over c if and only if

$$\forall \bar{x} \in [a,b]\ \forall [u,v] \subseteq [a,b]\ \exists \bar{x}' \in [u,v]\ (\bar{x}, c) \equiv (\bar{x}', c).$$

Proof. Suppose M is recursively saturated, and fix $c \in M$. One direction is obvious. For the other, let $[a,b] \in \mathcal{B}$ be a finite pregeneric interval, $x_1, x_2, \ldots, x_n \in [a,b]$, and $[u,v] \subseteq [a,b]$. Without loss of generality, assume that $n > 2$ and $x_1 < x_2 < \cdots < x_n$. Using pregenericity, let $x_1', x_n' \in [u,v]$ such that $(x_1, x_n, c) \equiv (x_1', x_n', c)$. By considering the recursive type

$$p(v_2, v_3, \ldots, v_{n-1}) = \{\phi(x_1', v_2, v_3, \ldots, v_{n-1}, x_n') : \phi(\bar{v}) \in \mathscr{L}_{\mathrm{A}} \text{ with } M \models \phi(\bar{x})\},$$

one can easily find $x_2', x_3', \ldots, x_{n-1}' \in M$ such that $(\bar{x}, c) \equiv (\bar{x}', c)$ using recursive saturation. In particular, $u \leqslant x_1' < x_2' < \cdots < x_{n-1}' < x_n' \leqslant v$, as required. □

Remark. The above argument also shows that modulo recursive saturation, pregenericity of a finite $\mathcal{B}$-interval $[a,b]$ over an element c in M is equivalent to

$$\forall [u,v] \subseteq [a,b]\ \exists a', b' \in [u,v]\ (a,b,c) \equiv (a',b',c).$$

Another way to strengthen the notion of pregenericity is to require an interval to be pregeneric over all elements in a cut I. In some very particular cases, this is possible.

Proposition 5.8. Suppose M is recursively saturated and $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$. Let $c, c' \in M$ and $[a,b]$ be a finite elementary interval such that $c, c' \ll a$ and $\mathrm{tp}(c) = \mathrm{tp}(c')$. Then

$$\forall [u,v] \subseteq [a,b]\ \exists a', b' \in [u,v]\ (a,b,c) \equiv (a',b',c').$$

In particular, if $c = c'$, then $[a,b]$ is pregeneric over c.

Proof. Suppose M is recursively saturated and $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$. Let $[a,b]$ be a finite elementary interval, $c, c' \ll a$ and $[u,v] \subseteq [a,b]$.

First, we find $a' > u$ with $(a, c) \equiv (a', c')$ and $a' \ll v$. Consider the recursive type

$$p(x) = \{\phi(x, c') \leftrightarrow \phi(a, c) : \phi(x, y) \in \mathscr{L}_{\mathrm{A}}\} \\ \cup \{t_n(x) < v : n \in \mathbb{N}\} \cup \{u < x\}.$$

Take $n \in \mathbb{N}$ and $\phi(x, y) \in \mathscr{L}_{\mathrm{A}}$ such that $M \models \phi(a, c)$. Using Proposition 3.1, pick an elementary cut I in $[u, v]$. Since $c \ll a$, we see that $M \models \mathsf{Q}x\ \phi(x, c)$. Our hypothesis on c and c' then implies that $M \models \mathsf{Q}x\ \phi(x, c')$. By the elementarity of I in M, we have $M \models \mathsf{Q}x \in I\ \phi(x, c')$. In particular,

$$M \models \exists x > u\ (t_n(x) < v \wedge \phi(x, c')).$$

Hence, $p(x)$ is finitely satisfied in M. Using recursive saturation, let $a' \in M$ realize $p(x)$, so that

$$(a, c) \equiv (a', c') \text{ and } u < a' \ll v. \tag{$*$}$$

Next, consider the recursive type

$$q(y) = \{\theta(a, b, c) \leftrightarrow \theta(a', y, c') : \theta(x, y, z) \in \mathscr{L}_{\mathrm{A}}\} \cup \{y < v\}.$$

Let $\theta(x, y, z) \in \mathscr{L}_{\mathrm{A}}$ such that $M \models \theta(a, b, c)$. We need to show $M \models \exists y < v\ \theta(a', y, c')$. Since $M \models \exists y\, \theta(a, y, c)$, we have $M \models \exists y\, \theta(a', y, c')$ by $(*)$. Thus

$$(\mu y)(\theta(a', y, c')) \in \mathrm{cl}(a', c') \subseteq M(\langle a', c' \rangle) < v,$$

proving that $q(y)$ is finitely satisfied in M. Using recursive saturation again, let b' realize $q(y)$ in M. Then

$$(a, b, c) \equiv (a', b', c') \text{ and } u < a' < b' < v,$$

as required. □

This does not always work.

Proposition 5.9. For every $B > \mathbb{N}$, there exists cofinally many $Y \in M$ such that, for every $\mathcal{B}$-interval $[a, b] \subseteq M_{<B}$ and every $d > \mathbb{N}$, there exists a nonstandard $c < d$ with $[a, b]$ not pregeneric over $\langle c, Y \rangle$.

Proof. Let $B > \mathbb{N}$. Using Proposition 2.1, let $Y \in M$ be a monotone indicator for $\mathcal{B}$ below a sufficiently large number bigger than B. Note that we can make the code Y arbitrarily large in this way.

Let $[a, b] \subseteq M_{<B}$ be a $\mathcal{B}$-interval and $d > \mathbb{N}$. Without loss of generality, suppose $Y(a, b) > d$. Using Lemma 2.7, pick $[u, v] \subseteq [a, b]$ such that $\mathbb{N} < Y(u, v) < d$. Let $c = Y(u, v)$. Then for all $\lceil a', b' \rfloor \subseteq [u, v]$, we have

$$Y(a', b') \leqslant Y(u, v) = c$$

by the monotonicity of Y, and

$$Y(a, b) > d > Y(u, v) = c.$$

Hence $(a, b, \langle c, Y \rangle) \not\equiv (a', b', \langle c, Y \rangle)$ for every $\lceil a', b' \rfloor \subseteq [u, v]$. Therefore, $[a, b]$ is not pregeneric over $\langle c, Y \rangle$. $\square$

These show that pregenericity is a stable and optimal notion. Another evidence of this is its relationship with arithmetic saturation.

Proposition 5.10. If for every $f \in M$, there are $B \in M$ and an indicator Y for $\mathcal{B}$ below B such that a pregeneric interval over $\langle f, Y \rangle$ exists in $M_{<B}$, then $\mathbb{N}$ is strong in M.

Proof. Suppose the hypothesis in the proposition holds. Let $f \colon \mathbb{N} \to M$ be a coded function in M. Abusing notation, we let f be a code for this function in M. Using the hypothesis, let $B \in M$ and Y be an indicator for $\mathcal{B}$ below B, and pick a $\mathcal{B}$-interval $[a, b] \subseteq M_{<B}$ that is pregeneric over $\langle f, Y \rangle$. Note that by the proof of Proposition 2.1, we may assume Y to be monotone.

We claim that $f(n) > \mathbb{N}$ if and only if $f(n) > Y(a, b)$ for all $n \in \mathbb{N}$. Note that since $[a, b] \in \mathcal{B}$, the "if part" is obvious. So let $n \in \mathbb{N}$ such that $f(n) > \mathbb{N}$. Using Lemma 2.7, let $[u, v] \subseteq [a, b]$ such that $\mathbb{N} < Y(u, v) < f(n)$. Recalling that $[a, b]$ is pregeneric over $\langle f, Y \rangle$, let $a', b' \in [u, v]$ such that

$$(a, b, \langle f, Y \rangle) \equiv (a', b', \langle f, Y \rangle). \tag{†}$$

By the monotonicity of Y, we have $Y(a', b') \leqslant Y(u, v) < f(n)$. Thus by (†), we get $Y(a, b) < f(n)$ as required. □

It may seem that the choice of the above collection of propositions is random, and the details in many of them are technical in nature. Actually, the overall aim of all these is to gather enough information to get a big picture of the fine structure of recursively saturated models of PA. This aim is not quite reached yet. This is one of the reasons why we currently still have more questions than theorems. I believe there are nice results to be explored in this direction. There will be more bits and pieces in the next chapter and in the final chapter.

While pregeneric intervals are interesting in their own right, the original reason for their introduction is to define *generic cuts* as their name implies.

Definition. A cut I is *generic (for $\mathcal{B}$)* if and only if it is contained in a pregeneric $\mathcal{B}$-interval over c for every $c \in M$.

Before saying anything deep about generic cuts, we need to check that these cuts actually lie in our universe of discourse. It makes use of the following easy consequence of axiom (2) for notions of intervals.

Fact 5.11. If $c \in M$ and $[a, b]$ is a $\mathcal{B}$-interval containing some element in $\mathrm{cl}(c)$, then $[a, b]$ is not constant over c. □

Lemma 5.12. All generic cuts for $\mathcal{B}$ are in $\mathscr{X}_{\mathcal{B}}$.

Proof. Let I be a generic cut for $\mathcal{B}$ and $a, b \in M$ such that $a \in I < b$. Using the genericity of I, let $[u, v]$ be a pregeneric interval over $\langle a, b \rangle$ containing I. By Fact 5.11, we know that $a, b \notin [u, v]$. So $\lceil a, b \rfloor \supseteq [u, v]$ is a $\mathcal{B}$-interval by axiom (3) for a notion of intervals. $\square$

Using Theorem 5.4 and Baire's Theorem, it is then straightforward to show that generic cuts exist.

Theorem 5.13. If M is countable and arithmetically saturated, then genericity is an enforceable property of $\mathcal{Z}$-cuts.

Proof. Suppose M is countable and arithmetically saturated. We play a Banach–Mazur game on $\mathcal{B}$. If $c \in M$, then $\exists$ can make the outcome of a play be contained in a pregeneric interval over c using Theorem 5.4 in just one step. Since M is countable and $\exists$ has ω many steps to play, she can ensure that the outcome is contained in a pregeneric interval over c for every $c \in M$. In other words, genericity is enforceable. $\square$

A direct consequence of Proposition 5.10 and the definition of genericity is that the strength of $\mathbb{N}$ in the hypothesis of the above theorem is necessary.

Corollary 5.14. If M contains a generic cut for $\mathcal{B}$, then $\mathbb{N}$ is strong in M. $\square$

CHAPTER 6

CONJUGACY OF NEIGHBOURING GENERIC CUTS

> After the celebrated work on models of PA, done in mid-seventies, mainly by Kirby and Paris, it was soon noticed that countable recursively saturated models for this theory arise very naturally. Roughly speaking each construction of a model for PA either gives directly a recursively saturated model (for example the Arithmetized Completeness Theorem) or an inessential variant of the construction yields a recursively saturated model (e.g., the indicator construction). Thus countable recursively saturated models of PA are natural objects to study, and so are their automorphism groups. The obvious problems here are: (i) how does Aut(M) act on M, and (ii) what are the properties of Aut(M) (either as an abstract group or as a topological group).
>
> Henryk Kotlarski (1995)
> Automorphisms of countable recursively saturated models of PA: a survey

First of all, we need to say what *conjugacy* means in the chapter title.

Definition. Let $c \in M$ and $I_1, I_2 \in \mathcal{C}$. We say that I_1 and I_2 are *conjugate over* c if and only if there is an automorphism $g \in \mathrm{Aut}(M, c)$ such that $I_1^g = I_2$. Two cuts are *conjugate* if and only if they are conjugate over 0.

Many families of cuts, mainly elementary ones, are known to behave very well under the action of automorphisms of the model. The first of these was discovered via *sequences of gaps* by Craig Smoryński (1982b). Other such families were studied later by Roman Kossak (1995), with stronger conjugacy properties considered. These include cuts of the form $M[b]$ in the species of elementary cuts where b realizes a minimal type.

In GCMA (Kaye 2008), Kaye proves that generic cuts have nice conjugacy properties as well. This chapter generalizes the results in GCMA to the current setting, including the back-and-forth system introduced there.

As in the previous chapter, we work in a fixed notion of intervals $\mathcal{B}$ and its corresponding species $\mathcal{Z} = \mathscr{Z}_{\mathcal{B}}$ throughout this chapter. Additionally, we assume countability and arithmetic saturation of our model M.

Theorem 6.1. Let $c \in M$ and $[a, b] \in \mathcal{B}$ be a pregeneric interval over c. Then any two generic cuts contained in $[a, b]$ are conjugate over c.

Proof. We use a back-and-forth argument.

Let $c \in M$ and $[a, b] \in \mathcal{B}$ be a pregeneric interval over c. Pick two generic cuts I and I' in $[a, b]$. At any stage of the back-and-forth, we have

- an interval $[u, v]$ containing I,
- an interval $[u', v']$ containing I', and
- tuples $\bar{r}, \bar{r}' \in M$

such that

- $[u, v]$ is pregeneric over $\langle c, \bar{r}\rangle$,
- $[u', v']$ is pregeneric over $\langle c, \bar{r}'\rangle$, and
- $(u, v, c, \bar{r}) \equiv (u', v', c, \bar{r}')$.

We show how to add an arbitrary ${}^*r$ to $\bar{r}$. In the process, we find ${}^*u, {}^*v$ to replace u, v and choose corresponding ${}^*u', {}^*v', {}^*r'$ while keeping $\bar{r}'$ unchanged. This constitutes the "forth" step. The "back" step is similar.

Using the definition of generic cuts, choose an interval $[{}^*u, {}^*v]$ that contains I and is pregeneric over $\langle u, v, c, \bar{r}, {}^*r\rangle$. Pick an automorphism $g \in \mathrm{Aut}(M, c)$ such that $\langle u, v, \bar{r}\rangle g = \langle u', v', \bar{r}'\rangle$, which is possible since $(u, v, c, \bar{r}) \equiv (u', v', c, \bar{r}')$ and M is recursively saturated.

It follows that $[{}^*ug, {}^*vg] \subseteq [u', v']$. Using the pregenericity of $[u', v']$ and recursive saturation, let $h \in \mathrm{Aut}(M, c, \bar{r}')$ such that $[u', v']^h \subseteq [{}^*ug, {}^*vg]$. The back-and-forth construction then continues by setting

$$[{}^*u', {}^*v'] = [{}^*ugh^{-1}, {}^*vgh^{-1}] \text{ and } {}^*r' = {}^*rgh^{-1}.$$

The required isomorphism is given by $\bar{r} \mapsto \bar{r}'$ at the end. □

This theorem reveals many genericity properties of generic cuts.

Corollary 6.2. (a) Any set of generic cuts that is dense in $\mathcal{Z}$ and is closed under the action of $\mathrm{Aut}(M)$ contains all generic cuts.

(b) A generic cut satisfies all enforceable properties that are invariant under the action of $\mathrm{Aut}(M)$.

(c) The set of generic cuts is the smallest enforceable subset of $\mathcal{Z}$ that is closed under the action of $\mathrm{Aut}(M)$.

Proof. Part (a) follows immediately from Theorem 6.1, and part (c) is a direct consequence of (b). So we only need to prove (b).

Let $\mathcal{G}$ be the set of all generic cuts and $\mathcal{P} \subseteq \mathcal{Z}$ be enforceable. By Theorem 5.13 and Fact 4.3(b), $\mathcal{G} \cap \mathcal{P}$ is enforceable. So the Baire Category Theorem and part (a) imply that $\mathcal{G} \cap \mathcal{P} = \mathcal{G}$. Hence $\mathcal{G} \subseteq \mathcal{P}$ as required. □

Theorem 6.1 says that the surroundings of a generic cut is blurred in a certain sense, which agrees with the fact that pregeneric intervals are homogeneous in nature. This blurry nature actually characterizes genericity. Note that comeagre-ness comes for free, which may justify why we consider comeagre sets rather than dense sets in the first place.

Corollary 6.3. The set of generic cuts is the unique subset $\mathcal{G}$ of $\mathcal{Z}$ such that

(i) $\mathcal{G}$ is a dense subset of $\mathcal{Z}$ that is closed under the action of $\mathrm{Aut}(M)$; and

(ii) for all $I \in \mathcal{G}$ and all $c \in M$, there is an interval $[a,b]$ containing I in which all cuts in $\mathcal{G}$ are conjugate to I over c.

Proof. Clearly, the definition of genericity together with Theorems 5.13 and 6.1 imply that the class of generic cuts satisfies the above conditions.

Let $\mathcal{G}$ denote the set of all generic cuts, and $\mathcal{G}' \subseteq \mathcal{Z}$ satisfy the above conditions. By Corollary 6.2(a), it suffices to prove $\mathcal{G}' \subseteq \mathcal{G}$.

Let $I \in \mathcal{G}'$ be arbitrary, and fix $c \in M$. Using condition (ii), let $[a,b]$ be an interval around I in which all $\mathcal{G}'$-cuts are conjugate over c. Using Theorem 5.4, let $[u,v] \subseteq [a,b]$ that is pregeneric over c. Take a $\mathcal{G}'$-cut J in $[u,v]$, which is possible by the density of $\mathcal{G}'$ in $\mathcal{Z}$. By our choice of $[a,b]$, the cuts I and J are conjugate over c. Let $g \in \mathrm{Aut}(M,c)$ such that $J^g = I$. Then $[u,v]^g \cap [a,b]$ is a pregeneric interval over c containing I. Since this can be done for every $c \in M$, the cut I is generic, as required. □

It is natural to ask exactly how large this blurry zone is. The following shows that one can improve Theorem 6.1 slightly.

Corollary 6.4. If $c \in M$ and $[a,b]$ is an interval satisfying

$$\exists x \in [a,b]\ \forall [u,v] \subseteq [a,b]\ \exists x' \in [u,v]\ (x,c) \equiv (x',c),$$

then all generic cuts in $[a,b]$ are conjugate over c.

Proof. Let $[a,b]$ be an interval and $x, c \in M$ such that

$$\forall [u,v] \subseteq [a,b]\ \exists x' \in [u,v]\ (x,c) \equiv (x',c). \qquad (*)$$

Pick two generic cuts I_1 and I_2 from $[a,b]$. Using the definition of generic cuts, let $[u_1,v_1]$ and $[u_2,v_2]$ be pregeneric intervals over $\langle a,b,c\rangle$ that contain I_1 and I_2 respectively. Note that $[u_1,v_1]$ and $[u_2,v_2]$ have to be subintervals of $[a,b]$.

Our plan is to map I_1 close enough to I_2 via x so that Theorem 6.1 can be applied. Using axiom (2) for a notion of intervals, let $[u_2',v_2']$ be a pregeneric subinterval of $[u_2,v_2]$

over c containing I_2 such that

$$u_2 \ll u_2' \ll v_2' \ll v_2. \tag{†}$$

Using $(*)$ and recursive saturation, let $g_1, g_2 \in \mathrm{Aut}(M, c)$ such that $xg_1 \in [u_1, v_1]$ and $xg_2 \in [u_2', v_2']$. It follows from (†) that $[u_1, v_1]^{g_1^{-1}g_2} \cap [u_2, v_2] \in \mathcal{B}$. By Theorem 6.1, both $I_1^{g_1^{-1}g_2}$ and I_2 are conjugate over c to the generic cuts in this intersection. Therefore, $(M, I_1, c) \cong (M, I_2, c)$. □

This turns out to be the best possible.

Proposition 6.5. Let $[a, b]$ be a $\mathcal{B}$-interval, $c \in M$, and $\mathcal{D}$ be a dense set of $\mathcal{Z}$-cuts in $[a, b]$. If all $\mathcal{D}$-cuts in $[a, b]$ are conjugate over c, then

$$\exists x \in [a, b]\ \forall [u, v] \subseteq [a, b]\ \exists x' \in [u, v]\ (x, c) \equiv (x', c).$$

Proof. Let $[a, b] \in \mathcal{B}$ and $\mathcal{D}$ be a dense set of $\mathcal{Z}$-cuts in $[a, b]$. Take $c \in M$ such that all $\mathcal{D}$-cuts in $[a, b]$ are conjugate over c. Using Theorem 5.4, let $[r, s] \subseteq [a, b]$ be a pregeneric interval of c, and pick $x \in [r, s]$. We show that this x works.

Let $[u, v] \subseteq [a, b]$ be arbitrary. We apply a similar trick as in the previous proof again. Using axiom (2) for a notion of intervals, let $[u', v']$ be a subinterval of $[u, v]$ such that

$$u \ll u' \ll v' \ll v.$$

Using the density of $\mathcal{D}$ in $[a, b]$, take $\mathcal{D}$-cuts $I \in [r, s]$ and $J \in [u', v']$. By assumption, I is conjugate to J over c. Let $h \in \mathrm{Aut}(M, c)$ such that $I^h = J$. Then $[r, s]^h \cap [u, v]$ is an interval whose preimage under h is a subinterval of $[r, s]$. Let $[r', s']$ be this preimage. Recall that $[r, s]$ is a pregeneric interval over c. So there exists an automorphism $g \in \mathrm{Aut}(M, c)$ such that $xg \in [r', s']$ and hence $xgh \in [u, v]$. □

As suggested by Kossak (1995), we can try to generalize Theorem 6.1 in another way.

Question 6.6. Let $I_1, I_2, J_1, J_2 \in \mathcal{Z}$ be generic such that $I_1 < I_2$ and $J_1 < J_2$. Suppose in addition that $c \in M$ and $[a, b] \in \mathcal{B}$ is a pregeneric interval over c containing I_1, I_2, J_1, J_2. Is it always true that

$$(M, I_1, I_2, c) \cong (M, J_1, J_2, c)?$$

What if there are more than two pairs of generic cuts?

Results in Chapter 7 may be helpful in tackling this problem, but we are unable to resolve it yet.

We end this chapter by looking at what conjugacy properties can tell us about genericity. Our main tool is Corollary 6.2(b), which tells us that generic cuts possess all enforceable properties invariant under automorphisms. For example, this implies that generic cuts are neither ω-limits nor regular cuts. Generic cuts are semiregular if and only if $\mathcal{B}$ is relatively indestructible. We also know that (M, I) cannot be recursively saturated if I is a generic cut because $\mathbb{N}$ is definable with parameters in (M, I). This final point leads us to a better understanding of the how complicated the notion of genericity is via a theorem on *chronic resplendency.*

Definition. A structure N is *chronically resplendent* if and only if it is resplendent, and for every $\bar{a} \in N$ and every formula $\Phi(\bar{x}, \bar{X})$ with second order variables $\bar{X}$, if $N \models \exists \bar{X}\ \Phi(\bar{a}, \bar{X})$, then there is an $\bar{A} \subseteq N$ such that $N \models \Phi(\bar{a}, \bar{A})$ and $(N, \bar{A})$ is resplendent in the expanded language.

Theorem 6.7. A countable recursively saturated structure is chronically resplendent.

Proof. Please see Theorem 15.8 in Kaye (1991). □

Proposition 6.8. There are no Σ^1_1 formula $\Phi(\bar{x}, X)$ and parameters $\bar{a} \in M$ such that I is a generic cut for $\mathcal{B}$ if and only if $M \models \Phi(\bar{a}, I)$ for every $I \subseteq M$.

Proof. Let $\Phi(\bar{x}, X)$ be a Σ^1_1 formula and $\bar{a} \in M$ such that I is a generic cut for $\mathcal{B}$ if and only if $M \models \Phi(\bar{a}, I)$ for every $I \subseteq M$. By Theorem 5.13, we know that generic cuts exist, and so $M \models \exists I\ \Phi(\bar{a}, I)$. Note that since M is countable and recursively saturated, M is chronically resplendent by Theorem 6.7. Using chronic resplendency, let $I \subseteq M$ such that $M \models \Phi(\bar{a}, I)$ and (M, I) is resplendent. It follows that (M, I) is recursively saturated in the expanded language. However, our choice of $\Phi(\bar{x}, X)$ and $\bar{a}$ implies that I is a generic cut for $\mathcal{B}$. This contradicts Corollaries 4.7 and 6.2(b). $\square$

A few characterizations for genericity are given in Corollaries 6.2 and 6.3. One may also want to simplify the definition of genericity.

Question 6.9. Call $I \in \mathcal{Z}$ a *constant cut* if and only if it is contained in a constant interval over c for every $c \in M$. It is easy to see that every generic cut is constant, and every constant cut is not an ω-limit. Do any of the these implications reverse?

CHAPTER 7

FURTHER CONJUGACY PROPERTIES AND QUANTIFIER ELIMINATION

> Abraham Robinson urged a different approach [to quantifier elimination]. For many interesting theories T, we have a mass of good structural information about the models of T: for example decomposition theorems, facts about algebraic or other closures of sets of elements, or results about embedding one model in another. It's hard to use these facts in an argument which concentrates on deducibility from T in the first-order predicate calculus.
>
> So Robinson's message was: to prove quantifier elimination, use model theory rather than syntax, when you can fit the model theory onto the known algebraic structure theory.
>
> Wilfrid Hodges (1993)
> **Model Theory**, §8.4, p. 381

In this chapter, we start to prove some new results that have no counterparts in GCMA. The main theorem in this chapter is a syntactic characterization of conjugacy for generic cuts. As a corollary, we obtain a description of the orbits of M under the action of $\mathrm{Aut}(M, I)$ where I is a generic cut. There is only one theorem with a similar flavour that we can find in the literature. So we reproduce it here for completeness.

Definition. Let $u \in M$. Define the function $2^u_\cdot \colon M \to M$ recursively by $2^u_0 = u$ and $2^u_{n+1} = 2^{2^u_n}$ for every $n \in M$. Then $\exp(u)$ is defined to be the smallest cut in M containing u that is closed under exponentiation. In other words,

$$\exp(u) = \sup\{2^u_n : n \in \mathbb{N}\}.$$

Definition. Let $I \in \mathcal{C}$ and $\bar{c}, \bar{c}' \in M$. We write $(\bar{c}, I) \equiv (\bar{c}', I)$ to mean $\bar{c}$ and $\bar{c}'$ are of the same length, and

$$(M, I) \models \varphi(\bar{c}) \leftrightarrow \varphi(\bar{c}')$$

for all $\mathscr{L}_{\mathrm{A}}^*$ formulas $\varphi(\bar{x})$.

Theorem 7.1 (Smith). Let $I \in \mathcal{C}$ and $a, b, c, u \in M$ such that $u > I$. If $(a, c, r) \equiv (b, c, r)$ for every $r \in \exp(u)$, then $(a, c, I) \equiv (b, c, I)$.

Proof. This is a consequence of Theorem 3.13 in Smith (1989). □

Remark. This is in fact very similar to Lemma 9.1.

In this chapter, $\mathcal{B}$ is a fixed notion of intervals, and $\mathcal{Z} = \mathscr{Z}_{\mathcal{B}}$ is its corresponding complete species. Assume M is countable and arithmetically saturated.

Instead of proving the main theorem straightaway, we motivate its proof using an example about GCMA indicators. We aim to show that in such situations, there exist countably infinitely many mutually non-conjugate generic cuts.

To prove this, we need to know when two generic cuts are *not* conjugate. Our first attempt is to use points in Skolem closures. It is obvious that if two cuts are separated by a definable point, then they cannot be conjugate.

Proposition 7.2. Let $\mathcal{D}$ be a dense set in $\mathcal{Z}$, and suppose $\mathcal{B} = \mathcal{B}^Y$ for some GCMA indicator Y. If $M \not\models \mathrm{Th}(\mathbb{N})$, then there are at least countably infinitely many mutually non-conjugate $\mathcal{D}$-cuts that are contained in $\overline{\mathrm{cl}}(\varnothing)$.

Proof. Let $\mathcal{D}$ be a dense set of cuts in $\mathcal{Z}$, and Y be a GCMA indicator such that $\mathcal{B} = \mathcal{B}^Y$. Suppose $M \not\models \mathrm{Th}(\mathbb{N})$.

Note that $M(0)$ exists and

$$M(0) = \sup\{(\mu y)(Y(0, y) \geqslant n) : n \in \mathbb{N}\} \subseteq_{\mathrm{e}} \overline{\mathrm{cl}}(0).$$

So $M(0) \subsetneq_{\mathrm{e}} \overline{\mathrm{cl}}(\varnothing)$ by Lemma 3.4. Take $a \in \mathrm{cl}(\varnothing)$ such that $a > M(0)$. Then $[0, a] \in \mathcal{B}$ by the definition of $M(0)$. Using the argument in the proof of Proposition 2.4 in GCMA, one can divide the $\mathcal{B}$-interval $[0, a]$ indefinitely into smaller subintervals by definable points. Since $\mathcal{D}$ is dense in $\mathcal{Z}$, we get any finite number of mutually non-conjugate $\mathcal{D}$-cuts in $\overline{\mathrm{cl}}(0)$. $\square$

When $M \models \mathrm{Th}(\mathbb{N})$, this trick does not work because there is no nonstandard definable point. This time, we make use of a function H that grows like an ascending sequence of gaps. The cuts in consecutive gaps cannot be conjugate because the maximum w such that $H(w)$ is in the cut are all in different congruence classes modulo a sufficiently large natural number.

Technical lemma 7.3. Let Y be a GCMA indicator. If $M \models \forall x \exists y\, Y(x, y) \geqslant n$ for each $n \in \mathbb{N}$, then there is a strictly increasing function $H\colon M \to M$ definable in M without parameters such that

$$H(k) \ll_{\mathcal{B}^Y} H(k+1)$$

for all large enough $k \in M$.

Proof. Let Y be a GCMA indicator. Suppose $M \models \forall x \exists y\, Y(x, y) \geqslant n$ for each $n \in \mathbb{N}$.

If $M \models \forall n \forall x \exists y\, Y(x, y) \geqslant n$, then let H be the function defined recursively by

$$H(0) = 0 \wedge \forall z \left(H(z+1) = (\mu y)(Y(H(z), y) \geqslant z+1)\right).$$

If $M \models \exists n \exists x \forall y\, Y(x, y) < n$, then define H by

$$H(0) = 0 \wedge \forall z \left(H(z+1) = (\mu y)(Y(H(z), y) \geqslant n)\right),$$

where $n = (\max m)(\forall x \exists y\, Y(x, y) \geqslant m)$. $\square$

Proposition 7.4. Let $\mathcal{D}$ be a dense set of $\mathcal{Z}$-cuts, and Y be a GCMA indicator such that $\mathcal{B} = \mathcal{B}^Y$.

(a) If $M \not\models \forall x \exists y\, Y(x, y) \geqslant n$ for some $n \in \mathbb{N}$, then no $\mathcal{Z}$-cut can contain $\overline{\mathrm{cl}}(\varnothing)$.

(b) If $M \models \forall x \exists y\, Y(x,y) \geqslant n$ for all $n \in \mathbb{N}$, then there are at least countably infinitely many mutually non-conjugate $\mathcal{D}$-cuts containing $\overline{\mathrm{cl}}(\varnothing)$.

Proof. Let $\mathcal{D}$ be a dense set of $\mathcal{Z}$-cuts, and Y be a GCMA indicator such that $\mathcal{B} = \mathcal{B}^Y$.

(a) Take $n \in \mathbb{N}$ such that $M \models \exists x \forall y\, Y(x,y) < n$. Let $x^* = (\mu x)(\forall y\, Y(x,y) < n)$. Then $x^* \in \mathrm{cl}(\varnothing)$ and no $\mathcal{B}$-interval is above x^* because $n \in \mathbb{N}$. So, there cannot be any $\mathcal{Z}$-cut above $\overline{\mathrm{cl}}(\varnothing)$.

(b) Suppose $M \models \forall x \exists y\, Y(x,y) \geqslant n$ for each $n \in \mathbb{N}$. Let H be a fast growing function whose existence is guaranteed by Lemma 7.3. Pick $x > \overline{\mathrm{cl}}(\varnothing)$ such that

$$\big([H(x+k), H(x+k+1)]\big)_{k \in \mathbb{N}}$$

is a sequence of $\mathcal{B}$-intervals. This is possible by recursive saturation. Using the density of $\mathcal{D}$ in $\mathcal{Z}$, take a $\mathcal{D}$-cut $I_k \in [H(x+k), H(x+k+1)]$ for each $k \in \mathbb{N}$.

Let $i, j \in \mathbb{N}$ such that I_i and I_j are conjugate. Since

$$(\max w)(H(w) \in I_i) = x + i, \text{ and}$$
$$(\max w)(H(w) \in I_j) = x + j,$$

it can be verified that for all sufficiently large $m \in \mathbb{N}$,

$$i \equiv j \mod m.$$

Hence, we must have $i = j$. Therefore, the cuts in $(I_k)_{k \in \mathbb{N}}$ are mutually non-conjugate. □

Putting these altogether gives us what we aimed for. We shall state this using a new piece of terminology.

Definition. The *conjugacy class* of a cut I is the orbit of I under the action of $\mathrm{Aut}(M)$.

Corollary 7.5. If $\mathcal{B} = \mathcal{B}^Y$ for some GCMA indicator Y, then there are exactly countably infinitely many conjugacy classes of generic cuts in M.

Proof. Let Y be a GCMA indicator such that $\mathcal{B} = \mathcal{B}^Y$.

Recall that by Theorem 6.1, any two generic cuts contained in the same pregeneric interval are conjugate. Since M is countable, this implies that there can only be at most countably infinitely many conjugacy classes of generic cuts in M.

On the other hand, note that it is not possible to have $M \models \mathrm{Th}(\mathbb{N})$ and

$$M \models \exists x \forall y\, Y(x, y) < n \text{ for some } n \in \mathbb{N}$$

both true at the same time. Otherwise, the truth of $\exists x \exists y\, Y(x, y) \geqslant n$ in M for every $n \in \mathbb{N}$ then implies the existence of a nonstandard definable element. Therefore we are done by Propositions 7.2, 7.4, Theorem 5.13, and the Baire Category Theorem. □

Remark. By Theorem 6.1 and Proposition 5.8, we already knew there is exactly one conjugacy class of generic cuts for $\mathcal{B}^{\mathrm{elem}}$.

All the above non-conjugacy claims are proved using a sentence that is true in one structure but not in the other. One may ask whether we can find non-conjugate generic cuts that give the same $\mathscr{L}_{\mathrm{A}}^*$ theory. The following example suggests that this may not be possible.

Proposition 7.6. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$, and let I be a generic cut for $\mathcal{B}^{\mathrm{elem}}$. If $a, b \in I$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$, then $(M, I, a) \cong (M, I, b)$.

Proof. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$ and let $a, b \in I \in \mathcal{Z}$ such that I is generic and $\mathrm{tp}(a) = \mathrm{tp}(b)$. Using the genericity of I, let $[r, s]$ be a pregeneric interval over $\langle a, b\rangle$ that contains I. Then by Fact 5.11, we necessarily have $a, b \ll r$.

Using Proposition 5.8 and recursive saturation, let $g \in \mathrm{Aut}(M)$ such that

$$a = bg \text{ and } [r, s]^g \subseteq [r, s].$$

Let $J = I^g$. Then

$$J = I^g \in [r,s]^g \subseteq [r,s]$$

so that both I and J are generic cuts in $[r,s]$. However, $[r,s]$ is pregeneric over a by Proposition 5.8. So by Theorem 6.1, there is an automorphism $h \in \mathrm{Aut}(M,a)$ such that $J^h = I$ and thus

$$(M,I,b) \cong (M,I^g,bg) = (M,J,a) \cong (M,J^h,ah) = (M,I,a),$$

as required. □

From one point of view, this proposition says that the formula $x \in I$ tells us a lot about an element x when we are in the above situation. On the contrary, the formula $x \notin I$ is much weaker.

Proposition 7.7. Suppose that all $\mathcal{Z}$-cuts are closed under addition and multiplication. If I is a generic cut, $c \in M$ and $B > I$, then there are $d, d' \in M$ such that $I < d, d' < B$ and $(d,c) \equiv (d',c)$, but $(d,c,I) \not\equiv (d',c,I)$.

Proof. Suppose all $\mathcal{Z}$-cuts are closed under addition and multiplication. Let $I \in \mathcal{Z}$ be generic, $c \in M$ and $B > I$. Using the genericity of I, let $[a,b] \in \mathcal{B}$ be a pregeneric interval over $\langle c, B\rangle$ containing I.

By Proposition 4.5 and Corollary 6.2(b), $I \neq M[b]$. So by the maximality of $M[b]$ from Proposition 3.1, we have $I < M[b]$. Let $w \in M[b] \setminus I$. By Lemma 3.5, $M(w) \neq M[b]$. Take $z \in M[b] \setminus M(w)$ and let $d = \langle w, z\rangle$. Note that $M[b] \in \mathcal{Z}$ is closed under addition and multiplication, and thus $d \in M[b]$. So now, we have

$$a \in I < w \ll z < \langle w, z\rangle = d \in M[b] < b.$$

Using Theorem 5.13 and the Baire Category Theorem, pick a generic cut $J \in [w,z] \subseteq [a,b]$. Then I and J are conjugate over $\langle c, B\rangle$ by Theorem 6.1. Let $g \in \mathrm{Aut}(M, \langle c, B\rangle)$

such that $J^g = I$. Suppose $d' = dg$ so that $(d, c, B) \equiv (d', c, B)$. In particular, as $d < B$, we have $d' < B$ as well. Note also that since $J < d$, we have

$$I = J^g < dg = d'.$$

Let π_L be the Skolem function defined by

$$\forall p \left(\pi_L(p) = (\mu x)\big(\exists y(p = \langle x, y\rangle)\big)\right).$$

Then $\pi_L(d) = \pi_L(\langle w, z\rangle) = w > I$, but since $w \in J$, we have

$$\pi_L(d') = \pi_L(dg) = (\pi_L(d))g = wg \in J^g = I.$$

Therefore, $(d, c, I) \not\equiv (d', c, I)$. □

Again, the above proof makes use of an $\mathscr{L}_\mathrm{A}^*$ formula to prove non-conjugacy. It may not be surprising to the reader now that the $\mathscr{L}_\mathrm{A}^*$ theory determines the isomorphism type of (M, I) when I is generic (partially because we mentioned it at the beginning of the chapter already). The surprising thing is: the formulas used in the proof of Proposition 7.4 suffice. These formulas turn out to be important in the study of generic cuts. So we take some time to set up the notation properly.

Definition. Let $\phi(\bar{x}, y)$ be an $\mathscr{L}_\mathrm{A}$ formula, $I \in \mathcal{C}$ and $\bar{c} \in M$. We write $\nu^I_{\phi(\bar{x},y)}(\bar{c})\!\downarrow$ for the $\mathscr{L}_\mathrm{A}^*$ formula

$$\exists y \in I \ \big(\phi(\bar{c}, y) \wedge \forall y' \in I \ (y' > y \rightarrow \neg\phi(\bar{c}, y))\big).$$

The expression $\nu^I_{\phi(\bar{x},y)}(\bar{c})\!\uparrow$ is the negation of $\nu^I_{\phi(\bar{x},y)}(\bar{c})\!\downarrow$. Define

$$\nu^I_{\phi(\bar{x},y)}(\bar{c}) = \begin{cases} (\max y \in I)(\phi(\bar{c}, y)), & \text{if } \nu^I_{\phi(\bar{x},y)}(\bar{c})\!\downarrow; \\ 0, & \text{otherwise.} \end{cases}$$

The language obtained from $\mathscr{L}_{\mathrm{Sk}}$ by adding a new predicate

$$\nu^I_{\alpha(\bar{x},y)}(\bar{x})\downarrow$$

for each $\mathscr{L}_{\mathrm{A}}$ formula is called $\mathscr{L}_\nu$. Structures with signature $\mathscr{L}^*_{\mathrm{A}}$ are interpreted as $\mathscr{L}_\nu$ structures in the natural way.

These ν's are recognized as bad points for constant intervals, as the following strengthening of Fact 5.11 shows.

Lemma 7.8. Let $I \in \mathscr{Z}$ be generic. If $c \in M$, and $[a, b] \in \mathcal{B}$ is a constant interval over c that contains I, then $\nu^I_{\phi(x,y)}(c) < a$ for every $\mathscr{L}_{\mathrm{A}}$ formulas $\phi(x, y)$.

Proof. Let $I \in \mathscr{Z}$ be generic, $c \in M$, and $[a, b] \in \mathcal{B}$ be a constant interval over c that contains I. Fix an $\mathscr{L}_{\mathrm{A}}$ formula $\phi(x, y)$. By Fact 5.11, we know that $0 < a$. So suppose $M \models \nu^I_{\phi(x,y)}(c)\downarrow$. Let $A = \nu^I_{\phi(x,y)}(c) + 1 \in I$. Then $M(A) < I$ by Proposition 4.5 and Corollary 6.2(b). Let $B \in M$ such that $M(A) < B \in I$. If $A > a$, then $[A, B] \subseteq [a, b]$ and

$$M \models \nu^I_{\phi(x,y)}(c) \in [a, b] \wedge \phi(c, \nu^I_{\phi(x,y)}(c))$$

while $M \models \forall y \in [A, B]\ \neg\phi(c, y)$ by the maximality of $\nu^I_{\phi(x,y)}(c)$, which is a contradiction since $[a, b]$ is constant over c. Therefore, $\nu^I_{\phi(x,y)}(c) < A \leqslant a$. $\square$

Here is the main theorem of this chapter.

Theorem 7.9. Let $c \in M$ and $I, J \in \mathscr{Z}$ be generic. Then $(M, I, c) \cong (M, J, c)$ if and only if

$$M \models \nu^I_{\alpha(x,y)}(c)\downarrow \leftrightarrow \nu^J_{\alpha(x,y)}(c)\downarrow$$

for every $\mathscr{L}_{\mathrm{A}}$ formula $\alpha(x, y)$.

Proof. One direction is obvious. For the other direction, let $c \in M$ and $I, J \in \mathscr{Z}$ be generic such that $M \models \nu^I_{\alpha(x,y)}(c)\downarrow \leftrightarrow \nu^J_{\alpha(x,y)}(c)\downarrow$ for every $\mathscr{L}_{\mathrm{A}}$ formula $\alpha(x, y)$. Without loss of generality, assume $I < J$. Using the genericity of I and J, pick a pregeneric interval $[a, b]$ over c containing I, and a pregeneric interval $[u, v]$ over c containing J. By Proposition 4.5 and Corollary 6.2(b), $M(a) < I$. Let $A \in M$ such that $M(a) < A \in I < b$.

Consider the recursive type

$$p(y) = \{u \leqslant y \leqslant v\} \cup \{\alpha(c, y) \leftrightarrow \alpha(c, A) : \alpha(x, y) \in \mathscr{L}_{\mathrm{A}}\}.$$

We show that this is finitely satisfied in M. Let $\alpha(x, y) \in \mathscr{L}_{\mathrm{A}}$ such that $M \models \alpha(c, A)$. Now if $M \models \nu^I_{\alpha(x,y)}(c)\downarrow$, then by Lemma 7.8 and the maximality of $\nu^I_{\alpha(x,y)}(c)$, we have

$$a \ll A \leqslant \nu^I_{\alpha(x,y)}(c) < a,$$

which is a contradiction. So $M \models \nu^I_{\alpha(x,y)}(c)\uparrow$. By our hypothesis, we have $M \models \nu^J_{\alpha(x,y)}(c)\uparrow$. Note that $A \in I < J$ and $M \models \alpha(c, A)$. So $M \models \mathsf{Q}y \in J \;\; \alpha(c, y)$. In particular, there exists $y \in J$ such that $M \models y \geqslant u \wedge \alpha(c, y)$. Thus $M \models \exists y \in [u, v] \;\; \alpha(c, y)$, as required.

Let B realize $p(y)$ in M. By construction, $\mathrm{tp}(A, c) = \mathrm{tp}(B, c)$. Using recursive saturation of M, let $g \in \mathrm{Aut}(M, c)$ such that $Ag = B \in [u, v]$. Since $a \ll A \ll b$, the intersection $[a, b]^g \cap [u, v]$ is a $\mathcal{B}$-interval. Using Theorem 5.13 and the Baire Category Theorem, pick a generic cut J' in this interval. By Theorem 6.1, J is conjugate to J' over c, and $(J')^{g^{-1}}$ is conjugate to I over c. Therefore, I is conjugate to J over c. □

Apart from giving alternative proofs of Propositions 6.5 and 7.6 for generic cuts, this theorem also implies a weak quantifier elimination result.

Corollary 7.10. Let $I \in \mathscr{Z}$ be generic and $a, b \in M$. Then $(M, I, a) \cong (M, I, b)$ if and only if a and b satisfy the same quantifier free $\mathscr{L}_\nu$ formulas with respect to I. In particular, (M, I) is ω-homogeneous as an $\mathscr{L}^*_{\mathrm{Sk}}$ structure. □

We are not yet able to prove a *real* quantifier elimination result, i.e., we still cannot prove that every $\mathscr{L}^*_{\mathrm{A}}$ formula is uniformly equivalent to a *single* quantifier free $\mathscr{L}_\nu$ formula in (M, I). This is closely related to the question of whether (M, I) is recursively saturated for quantifier free $\mathscr{L}_\nu$ formulas. It is because by Corollary 4.7, if Φ is a set of $\mathscr{L}_{\mathrm{A}}^*$ formulas such that every $\mathscr{L}_{\mathrm{A}}^*$ formula is equivalent uniformly to a Φ formula in a way coded in M, then (M, I) cannot be Φ recursively saturated.

On the other hand, we do have a counterexample showing that the minimal candidate does not work in general. The idea is very similar to that in Proposition 7.4(b), but more specific and more powerful machineries are used.

Proposition 7.11. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$, and let $I \in \mathcal{Z}$ be generic. Then the formula

$$(\max j)((x)_j \in I) \text{ is even}$$

is not uniformly equivalent in (M, I) to a quantifier free $\mathscr{L}_{\mathrm{Sk}}^*$ formula. In fact, it is not even uniformly equivalent to an infinite conjunction of quantifier free $\mathscr{L}_{\mathrm{Sk}}^*$ formulas.

Proof. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$ and let $I \in \mathcal{Z}$ be generic. Using recursive saturation, let $c \in M$ code an ascending sequence of gaps of length ω, i.e., c codes a sequence of nonstandard length such that

$$(c)_i \ll (c)_{i+1}$$

for each $i \in \mathbb{N}$. Let $l \in M$ be the length of this sequence. Without loss of generality, assume this sequence is strictly increasing on its domain. Using Proposition 2.1, let Y be an indicator for $\mathcal{B}$ below $\max_{i<l}(c)_i + 1$. Then the function $f\colon \mathbb{N} \to M$ defined by

$$f(i) = Y((c)_i, (c)_{i+1})$$

for each $i \in \mathbb{N}$ is coded in M. Using the strength of $\mathbb{N}$ in M, let $\nu \in M$ be nonstandard such that for every $i \in \mathbb{N}$, we have

$$f(i) > \mathbb{N} \text{ iff } f(i) > \nu.$$

Note that since c codes an ascending sequence of gaps, $f(i) > \nu$ for all $i \in \mathbb{N}$. By overspill, let $m > \mathbb{N}$ such that

$$\forall i < m\ f(i) > \nu.$$

Using arithmetic saturation, let $i < m$ be nonstandard such that $i \notin \mathrm{cl}(c)$. Then we get

$$(c)_{i-1} \ll (c)_i \ll (c)_{i+1}$$

by our choice of ν.

Using Theorem 5.13, pick generic cuts $I \in [(c)_{i-1}, (c)_i]$ and $J \in [(c)_i, (c)_{i+1}]$. Notice that Proposition 5.8 and Theorem 6.1 imply that I and J are conjugate. Let $g \in \mathrm{Aut}(M)$ such that $I = J^g$ and set $d = cg$. Then by our choices of I and J,

$$(\max j)((c)_j \in I) \text{ and } (\max j)((c)_j \in J)$$

are of different parities. Hence

$$(c, I) \ncong (c, J) \cong (cg, J^g) = (d, I).$$

On the other hand, if t is a Skolem function such that $t(c) \in [(c)_{i-1}, (c)_{i+1}]$, then the c-definable point

$$(\mu j)((c)_j \geqslant t(c))$$

is either $i-1$, i or $i+1$, which is contradictory to our choice of i. So for every Skolem function t, we either have $t(c) < (c)_{i-1}$, or $(c)_{i+1} < t(c)$. It follows that

$$t(c) \in I \text{ iff } t(c) < (c)_{i-1} \text{ iff } t(c) \in J \text{ iff } t(cg) \in J^g \text{ iff } t(d) \in I$$

for every Skolem function t in $\mathscr{L}_{\mathrm{A}}$. Thus, c and d have the same quantifier free $\mathscr{L}^*_{\mathrm{Sk}}$ type as $cg = d$. Therefore, the formula

$$(\max j)((x)_j \in I) \text{ is even}$$

cannot be uniformly equivalent to any conjunction of quantifier free $\mathscr{L}^*_{\mathrm{Sk}}$ formulas. □

Question 7.12. Notice that the formula given by the above proposition is equivalent to

$$\exists w\big((x)_{2w} = \nu^I_{\exists j(y=(x)_j)}(x)\big).$$

Is it equivalent in (M, I) to a quantifier free $\mathscr{L}_\nu$ formula whenever I is generic for $\mathcal{B}$?

Question 7.13. Can one prove a similar proposition for a general notion of intervals?

CHAPTER 8

ELEMENTARY GENERIC CUTS

> Many notions and results have two distinct versions, a western version dealing with arbitrary formulas, and an eastern version dealing with quantifier-free formulas.
>
> H. Jerome Keisler (1977)
> Fundamentals of model theory, §1, p. 49

Elementary cuts in models of arithmetic are so thoroughly studied that we feel it necessary to devote a chapter just to gather the facts about generic cuts for the notion of elementary intervals from the literature and from previous chapters.

We work in the notion of intervals $\mathcal{B} = \mathcal{B}^{\text{elem}}$ and the complete species $\mathcal{Z} = \mathcal{Z}^{\text{elem}} = \mathscr{Z}_{\mathcal{B}}$ of elementary cuts in this chapter. Generic cuts in this species are call *elementary generic cuts*. We assume M to be countable and arithmetically saturated throughout the chapter. Fix an elementary generic cut I in M.

The study of elementary cuts started from the works of Smoryński (1981, 1982b) and Kotlarski (1983, 1984a) in the the early 1980s. As mentioned in Chapter 6, *sequences of gaps*, introduced by Smoryński and Stavi (1980), gave us first examples of nice elementary cuts. Elementary generic cuts are never limits of sequences of gaps of length ω. One can see this by looking at Proposition 4.4. Using the following theorem on ascending sequence of gaps, one can prove a weak form of Corollary 4.7 for elementary cuts.

Theorem 8.1 (Smoryński (1981, Theorem 2.8)). If J is a $\mathcal{Z}$-cut such that (M, J) is recursively saturated as an $\mathscr{L}_{\mathrm{A}}^{*}$ structure, then J is the limit of an ascending sequence of gaps of length J. $\square$

Proposition 8.2. The $\mathscr{L}_{\mathrm{A}}^*$ structure (M, I) is not recursively saturated.

Proof. Suppose (M, I) is recursively saturated. Using Theorem 8.1, let $c \in M$ code an ascending sequence of gaps of length I such that

$$\sup\{(c)_i : i \in I\} = I.$$

Using the genericity of I, pick a pregeneric interval $[a, b] \in \mathcal{B}$ over c containing I. Note that the sequence $((c)_i)_{i \in I}$ is cofinal in I. So let $i \in I$ such that $(c)_i > a$. By Theorems 5.13 and 6.1, I is conjugate to a generic cut in $[(c)_i, (c)_{i+1}] \subseteq [a, b]$ over c. This is impossible since no $\mathcal{Z}$-cut $J \in [(c)_i, (c)_{i+1}]$ can satisfy

$$\{(c)_j \in J : j \in M \text{ is less than the length of } c\} \subseteq_{\mathrm{cf}} J. \qquad \square$$

Here is a related question.

Question 8.3. What is

$$\{c \in M : \mathbb{N} \text{ is definable in } (M, I, c)\}?$$

In particular, is it a subset of $M \setminus I$?

All our known examples of elements $c \in M$ for which $\mathbb{N}$ is definable in (M, I, c) are above I.

In spite of Proposition 8.2, a theorem by Kotlarski together with Proposition 4.5 easily tell us that elementary generic cuts are recursively saturated as $\mathscr{L}_{\mathrm{A}}$ structures.

Theorem 8.4 (Kotlarski (1983, Lemmas 2 and 4)). An elementary cut J is recursively saturated if and only if $J \neq M(a)$ for any $a \in M$. $\square$

In particular, $I \cong M$ by standard results on recursively saturated models. It follows from the above theorem and Lemma 3.5 that there is an elementary end extension of I in which I is not generic.

Proposition 4.5 also gives us some information about automorphisms fixing I pointwise via a theorem by Kotlarski (1984a).

Theorem 8.5 (Kotlarski (1984a, Theorem 4.1)). Let J be an elementary cut in M. If $J \neq M[b]$ for any $b \in M$, then J is *closed in* M, i.e.,

$$\forall b > J \ \exists g \in \operatorname{Aut}(M) \ \big(\forall x \in J \ xg = x \ \text{ and } \ bg \neq b\big). \qquad \square$$

Corollary 8.6. All elementary generic cuts are closed. □

This brings us back to the topic of conjugacy properties. Exceptionally, all finite elementary intervals are pregeneric by Proposition 5.8. Another consequence of this result is Proposition 7.6, which says that

$$\forall a, b \in I \ \big(\operatorname{tp}(a) = \operatorname{tp}(b) \Rightarrow (M, I, a) \cong (M, I, b)\big).$$

This relates generic cuts to the notion of *free cuts* defined by Kossak.

Definition (Kossak (1986, 1995)). An elementary cut I is *free* if and only if whenever $a, b \in I$ with $\operatorname{tp}(a) = \operatorname{tp}(b)$, we have $(a, I) \equiv (b, I)$.

Corollary 8.7. All elementary generic cuts are free. □

This provides a new way to construct free cuts. By Theorem 6.1 and Proposition 5.8, all elementary generic cuts are conjugate, and hence by Theorem 5.13 and Fact 4.3(a), the orbit of I under the action of $\operatorname{Aut}(M)$ has cardinality $2^{\aleph_0}$. This partially answers a question by Kossak (1995, Problem 4.7). However, in view of the above discussion, this does not provide us with an example of a free cut I such that (M, I) is recursively saturated. Proposition 7.7 says something about the degree of freeness of I. In Kossak's (1986) terminology, it says that I is the largest initial segment J in M such that I is *J-free* in M.

On the other hand, using a theorem by Kossak and Bamber (1996), one can verify that all elements definable without parameters in (M, I) are in $\operatorname{cl}(\varnothing)$.

Theorem 8.8 (Kossak and Bamber (1996, Theorem 4.1)). If J is a cut in M that is closed under exponentiation, then every element definable in (M, J) without parameters is in $\mathrm{cl}(c)$ for some $c \in J$. □

Let us turn to quantifier elimination. It is remarked in the previous chapter that we are unable to find a real quantifier elimination result. In the context of elementary generic cuts, we are a little bolder and make the following conjecture.

Conjecture 8.9. Let $c \in M$ and $\theta(\bar{x}, y)$ be an $\mathscr{L}_{\mathrm{A}}$ formula. If $([r_i, s_i])_{i \leqslant n}$ is a sequence of $\mathcal{B}$-intervals containing I such that $[r_{i+1}, s_{i+1}]$ is pregeneric over $\langle c, r_0, \ldots, r_i, s_0, \ldots, s_i \rangle$ for every $i < n$, then

$$\begin{aligned} & (M, I) \models \forall x_0 \in I\ \exists x_1 \in I\ \cdots\ \mathbf{Q} x_n \in I\ \theta(\bar{x}, c) \\ \Leftrightarrow \quad & M \models \forall x_0 < r_0\ \exists x_1 < r_1\ \cdots\ \mathbf{Q} x_n < r_n\ \theta(\bar{x}, c). \end{aligned}$$

It is an easy exercise to show that this conjecture is true when $n = 2$.

There are other bits and pieces that we know about elementary generic cuts, mainly from Chapter 4. For example, by Propositions 4.9 and 4.10, I is semiregular but not regular in M.

Proposition 8.10. The notion of elementary intervals $\mathcal{B}^{\mathrm{elem}}$ is relatively indestructible.

Proof. Let $[a, b] \in \mathcal{B}$. Consider the recursive type

$$p(x) = \{\forall i < a\ (t_n((x)_i) < (x)_{i+1}) : n \in \mathbb{N}\} \cup \{(x)_0 = a \wedge (x)_a \leqslant b\}.$$

This is finitely satisfied in M since $[a, b]$ contains an elementary cut. Any element realizing $p(x)$ in M witnesses the relative indestructibility of $[a, b]$. □

CHAPTER 9

FURTHER DIRECTIONS

> In fact, an appreciation of beauty is essential, or at least very useful indeed, for solving problems [...]. Often, one can reject a line of attack on the grounds that it is simply not elegant enough to work. One may be wrong, but it is still efficient to look for beautiful solutions first, and settle for ugly ones only as a last resort. Similarly, finding the solution of a problem involves a great deal of rather vague reasoning, following hunches, making guesses and so on. How does one know whether one of these guesses will survive a later, more rigorous scrutiny? Well, one doesn't, [...] but it is a good rule of thumb that the more beautiful the guess, the more likely it is to survive.
>
> Timothy Gowers (2000)
> The importance of mathematics

An area related to generic cuts that is begging to be explored is about the automorphism group $\mathrm{Aut}(M, I)$ where I is generic for some notion of intervals. Theorem 6.1 and Corollary 7.10 provide new ways to construct automorphisms in this group. The back-and-forth system taken from GCMA, together with the well-known ones, including that described in the following lemma, suggest that the structure of such groups is quite rich. Although Lemma 9.1 does not seem to be new, we cannot find any reference to it in the literature. This gives us an excuse to include (part of) the full proof here.

Throughout this chapter, we suppose M to be countable and recursively saturated. We do not need $\mathbb{N}$ to be strong, but whenever we talk about a generic cut, we know that $\mathbb{N}$ has to be strong by Proposition 5.10. Denote the automorphism group $\mathrm{Aut}(M)$ by G.

Lemma 9.1. Let $u \in M$ and $I = \exp(u)$. Then for all $a > I$, the following are equivalent:

(1) there is an automorphism in $G_{(I)}$ moving a;

(2) there exists $b \in M$ distinct from a such that for each $\mathscr{L}_{\mathrm{A}}$ formulas $\theta(x, y)$ and for each $n \in \mathbb{N}$, we have

$$M \models \forall r < 2^u_n \, (\theta(a, r) \leftrightarrow \theta(b, r));$$

(3) $a \notin \mathrm{cl}(I)$, i.e., no finite tuple in I defines a in M.

Proof. Let $u \in M$ and $I = \exp(u)$. Fix $a > I$. We prove

$$(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1).$$

The implication (1) $\Rightarrow$ (3) is obvious.

Suppose (3) is true. Consider the recursive type

$$p(x) = \{\forall r < 2^u_n \, (\theta(a, r) \leftrightarrow \theta(x, r)) : n \in \mathbb{N} \text{ and } \theta(x, y) \in \mathscr{L}_{\mathrm{A}}\} \cup \{\neg a = x\}.$$

Let $\theta_1(x, y), \theta_2(x, y), \ldots, \theta_k(x, y) \in \mathscr{L}_{\mathrm{A}}$ and $n \in \mathbb{N}$. We want to show

$$M \models \exists x \neq a \bigwedge_{i=1}^{k} \big(\forall r < 2^u_n \, (\theta_i(a, r) \leftrightarrow \theta_i(x, r))\big). \tag{$*$}$$

For every $i \in \{1, 2, \ldots, k\}$, let $R_i = \{r < 2^u_n : M \models \theta_i(a, r)\} \subseteq M_{<2^u_n}$. So R_i is coded in M and

$$R_i \leqslant M_{<2^u_n} < 2^u_{n+1} \in \exp(u) = I$$

for each $i = 1, 2, \ldots, k$, because our coding apparatus is well-behaved. On the other hand, with a being a witness, we have

$$M \models \exists x \bigwedge_{i=1}^{k} \big((\forall r \in R_i \; \theta_i(x, r)) \wedge (\forall r \notin R_i \; \neg\theta_i(x, r))\big).$$

Let $a^* = (\mu x)\big(\bigwedge_{i=1}^{k} \big((\forall r \in R_i \; \theta_i(x, r)) \wedge (\forall r \notin R_i \; \neg\theta_i(x, r))\big)\big)$. Then $a^* \in \mathrm{cl}(\bar{R}, u) \subseteq \mathrm{cl}(I)$. In particular, $a^* \neq a$, and so $(*)$ is true.

Therefore, $p(x)$ is finitely satisfied in M. By recursive saturation, $p(x)$ is realized in

M. This is exactly what we need to prove (2).

The implication (2) $\Rightarrow$ (1) is the standard Kotlarski–Smoryński–Vencovská Lemma. The proof can be found, say, in Smoryński (1982a, Lemma 1.2). $\square$

There are many points of contact between automorphism groups related to a generic cut and those studied in the literature. We will state a few questions here, and leave it to the reader's imaginations to come up with others. For the rest of this chapter, we fix a cut I that is generic for a notion of intervals $\mathcal{B}$.

Question 9.2. Is $\mathrm{Aut}(M, I)$ a maximal subgroup of $\mathrm{Aut}(M)$?

Note that $\mathrm{Aut}(M, I)$ is naturally equipped with a topology, namely that generated by cosets of pointwise stabilizers of finite tuples from M.

Question 9.3. It is straightforward to see that $G_{(I)}$ is a closed normal subgroup of $G_{\{I\}}$. What are the other closed normal subgroups of $G_{\{I\}}$? In particular, if $M \models \mathrm{Th}(\mathbb{N})$, is $G_{(I)}$ the only closed normal subgroup of $G_{\{I\}}$?

Generic cuts are highly undefinable by their nature. For example, it follows from Corollary 4.7 that I is not definable in (M, g) for any recursively saturated automorphism g. One may suspect more to be true.

Conjecture 9.4. If $g \in \mathrm{Aut}(M)$ such that (M, g) is recursively saturated and g moves some point in I, then $I^g \neq I$.

Another topic that is worth looking into is about $\mathscr{L}_{\mathrm{A}}^*$ elementary extensions of the structure (M, I). By some standard model theoretic techniques, we know that there is a countable elementary extension of (M, I) that is recursively saturated in the expanded language. So genericity is not preserved in $\mathscr{L}_{\mathrm{A}}^*$ elementary extensions by Corollary 4.7. However, is there any proper elementary extension $(N, J) \succ (M, I)$ such that J is still generic in N? More generally, one may ask the following.

Question 9.5. Let $(N, J) \succ (M, I)$ be an $\mathscr{L}_{\mathrm{A}}^*$ elementary extension. Are there classes of $\mathscr{L}_{\mathrm{A}}^*$ formulas Φ^- and Φ^+ with $\Phi^- \subseteq \Phi^+$ such that J is generic for $\mathcal{B}$ in N whenever (N, J) is Φ^- recursively saturated but not Φ^+ recursively saturated?

There are other notions of extensions involving the structure (M, I). Nonregularity of I in M tells us about one of them, namely, there can be no I-extension of M.

Recall from Example 2.4 that each partial inductive satisfaction class gives rise to a notion of intervals. In view of the diverse variety of satisfaction classes, we expect the interactions between these notions of intervals to be quite different from those between our more familiar examples.

Finally, let us return to the structure theory of M that involves pregeneric intervals and generic cuts. A discussion of this appeared in Chapters 5 and 6 already. As mentioned there, an elegant theory is emerging from those scattered results. A coherent theory successfully developed in this direction will give us some deep understanding of countable arithmetically saturated models in an intuitive way.

We pursue this by attempting some of these problems posed in the previous chapters. On the way, we ask a few more questions.

Definition. Let $c \in M$. An element $x \in M$ is said to be *distinguished (for $\mathcal{B}$) over* $c \in M$ if and only if it is not contained in any constant interval over c. The element x is said to be *undistinguished (for $\mathcal{B}$) over* c if and only if it is not distinguished over c.

It is easy to see that if an element x is undistinguished over $c \in M$, then it is actually contained in a pregeneric interval over c. It can also be verified that if x is distinguished, and both $M(x)$ and $M[x]$ exist, then every element in $M(x) \setminus M[x]$ is distinguished.

Question 9.6. Are all distinguished elements over $c \in M$ *isolated over* c? In other words, is it true that if $x \in M$ is distinguished over c, then there exists a set $A \subseteq M$ definable with the parameter c such that either

- $x = \max(A)$ or $x = \min(A)$;
- $x \neq \max(A)$ and $x \ll \min(A_{>x})$; or
- $x \neq \min(A)$ and $\max(A_{<x}) \ll x$?

Note that all isolated elements are distinguished.

As mentioned in Chapter 5, a nice property of constant intervals is that as long as two constant intervals intersect in an interval, then their union is also constant. As a result, we can talk about the following.

Definition. Let $c \in M$. For an undistinguished element $x \in M$, define the *$\mathcal{B}$-community of x over c* to be the union of all constant intervals over c containing x.

Question 9.7. Let c be an element of M and $C \subseteq M$ be a $\mathcal{B}$-community over c. It can be verified that $\inf(C) \in \mathscr{Z}_{\mathcal{B}}$. Moreover, it is neither a generic cut nor of the form $M[b]$ for some $b \in M$. Is it a kind of the cuts that are already known to us? If yes, what is it? How about $\sup(C)$?

Knowledge in various kinds of types and interstices, say from Kossak and Schmerl (2006), will be useful in answering some of these questions.

LIST OF REFERENCES

> Where is the place of mathematics? Where does it exist? On the printed page, of course, and prior to printing, on tablets or on papyri. Here is a mathematical book—take it in your hand; you have a palpable record of mathematics as an intellectual endeavor. But first it must exist in people's minds, for a shelf of books doesn't create mathematics.
>
> Philip J. Davis and Reuben Hersh (1981)
> **The Mathematical Experience**, Chapter 1, p. 8

Brouwer, L.E.J. (1981) **Brouwer's Cambridge Lectures on Intuitionism**. Cambridge: Cambridge University Press. Materials from various series of lectures from 1946 to 1951 edited by D. van Dalen.

Davis, P.J. and Hersh, R. (1981) **The Mathematical Experience**. Boston: Birkhäuser. With an introduction by Gian-Carlo Rota.

Friedman, H. (1999) "Perfect" statements. **FOM e-mail List**. URL: `http://www.cs.nyu.edu/pipermail/fom/1999-February/002561.html` [Accessed 28 January 2007].

Goldblatt, R. (1984) **Topoi: The Categorial Analysis of Logic**, Studies in Logic and the Foundations of Mathematics, vol. 98, revised ed. Amsterdam: North-Holland Publishing Company.

Gowers, W.T. (2000) The importance of mathematics. Keynote address given at the Millennium Meeting of the Clay Mathematics Institute in Paris.

Hodges, W. (1993) **Model Theory**, Encyclopedia of Mathematics and its Application, vol. 42. Cambridge: Cambridge University Press.

Kaye, R. (1991) **Models of Peano Arithmetic**, Oxford Logic Guides, vol. 15. Oxford: Clarendon Press.

Kaye, R. (2007) **The Mathematics of Logic: A guide to completeness theorems and their applications**. Cambridge: Cambridge University Press.

Kaye, R. (2008) Generic cuts models of arithmetic. **Mathematical Logic Quarterly**, 54 (2): 129–144

Kaye, R. and Wong, T.L. (2007) On interpretations of arithmetic and set theory. **Notre Dame Journal of Formal Logic**, 48 (4): 497–510

Keisler, H.J. (1977) “Fundamentals of model theory.” In Barwise, J., Keisler, H.J., Kunen, K., Moschovakis, Y.N. and Troelstra, A.S. (eds.) **Handbook of Mathematical Logic**. Amsterdam: North-Holland Publishing Company. Studies in Logic and the Foundations of Mathematics, vol. 90, chap. A.2, pp. 47–103

Kirby, L., McAloon, K. and Murawski, R. (1981) Indicators, recursive saturation and expandability. **Fundamenta Mathematicae**, 114 (2): 127–139

Kirby, L.A.S. (1977) **Initial segments of models of arithmetic**. Ph.D. thesis, Manchester University.

Kirby, L.A.S. and Paris, J.B. (1977) “Initial segments of models of Peano’s axioms.” In Lachlan, A., Srebrny, M. and Zarach, A. (eds.) **Set Theory and Hierarchy Theory V: Bierutowice, Poland 1976**. Berlin: Springer-Verlag. Lecture Notes in Mathematics, vol. 619, pp. 211–226

Kossak, R. (1986) Remarks on free sets. **Bulletin of the Polish Academy of Sciences: Mathematics**, 34 (3–4): 117–122

Kossak, R. (1995) Four problems concerning recursively saturated models of arithmetic. **Notre Dame Journal of Formal Logic**, 36 (4): 519–530

Kossak, R. and Bamber, N. (1996) On two questions concerning the automorphism groups of countable recursively saturated models of PA. **Archive for Mathematical Logic**, 36 (1): 73–79

Kossak, R. and Schmerl, J.H. (2006) **The Structure of Models of Peano Arithmetic**, Oxford Logic Guides, vol. 50. Oxford: Clarendon Press, Oxford University Press.

Kotlarski, H. (1983) On elementary cuts in models of arithmetic. **Fundamenta Mathematicae**, 115 (1): 27–31

Kotlarski, H. (1984a) On elementary cuts in recursively saturated models of Peano arithmetic. **Fundamenta Mathematicae**, 120 (3): 205–222

Kotlarski, H. (1984b) Some remarks on initial segments in models of Peano arithmetic. **The Journal of Symbolic Logic**, 49 (3): 955–960

Kotlarski, H. (1995) Automorphisms of countable recursively saturated models of PA: a survey. **Notre Dame Journal of Formal Logic**, 36 (4): 505–518

Oxtoby, J.C. (1971) **Measure and Category: A Survey of the Analogies between Topological and Measure Spaces**, Graduate Texts in Mathematics, vol. 2. New York: Springer-Verlag.

Pacholski, L., Wierzejewski, J. and Wilkie, A.J. (eds.) (1980) **Model Thoery of Algebra and Arithmetic: Proceedings of the Conference on Applications of Logic to Algebra and Arithmetic Held at Karpacz, Poland, September 1–7, 1979**, Lecture Notes in Mathematics, vol. 834. Berlin: Springer-Verlag.

Paris, J.B. (1980) “A hierarchy of cuts in models of arithmetic.” In Pacholski, Wierzejewski and Wilkie (1980), pp. 312–337

Ramsey, F.P. (1930) On a problem of formal logic. **Proceedings of the London Mathematical Society**, s2-30 (1): 264–286

Smith, S.T. (1989) Extendible sets in Peano arithmetic. **Transactions of the American Mathematical Society**, 316 (1): 337–367

Smoryński, C. (1981) Elementary extensions of recursively saturated models of arithmetic. **Notre Dame Journal of Formal Logic**, 22 (3): 193–203

Smoryński, C. (1982a) “Back-and-forth inside a recursively saturated model of arithmetic.” In van Dalen, D., Lascar, D. and Smiley, T.J. (eds.) **Logic Colloquium ’80: Papers intended for the European Summer Meeting of the Association for Symbolic Logic**. Amsterdam: North-Holland Publishing Company. Studies in Logic and the Foundations of Mathematics, vol. 108, pp. 273–278

Smoryński, C. (1982b) A note on initial segment constructions in recursively saturated models of arithmetic. **Notre Dame Journal of Formal Logic**, 23 (4): 393–408

Smoryński, C. and Stavi, J. (1980) “Cofinal extension preserves recursive saturation.” In Pacholski et al. (1980), pp. 338–345