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Models of the Weak König Lemma

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Abstract

This paper surveys the model-theoretic aspects of two topics initiated by Kazuyuki Tanaka related to the Weak König Lemma: self-embedding, and conservativity for sentences of the form $\forall X\ \exists! Y\ \theta(X,Y)$, where $\theta(X,Y)$ is an arithmetical formula. It includes a few recent developments and proof sketches.

Key words: Weak König Lemma, Kazuyuki Tanaka, model theory of arithmetic

For historical reasons, the model theory of fragments of Peano arithmetic is more thoroughly studied than its second-order counterpart. Nevertheless, the connections with reverse mathematics and recursion theory have made subsystems of second-order arithmetic more popular nowadays. Kazuyuki Tanaka plays an important role in linking the model-theoretic tradition with second-order arithmetic, especially in relation to the Weak König Lemma. This paper surveys several pieces of recent research originated from this contribution of Tanaka's.

After fixing some notation and conventions, we describe in Section 1 the model-theoretic content of WKL_0^* in terms of subsets coded in end extensions. This will be key in subsequent arguments. We then devote Section 2 to Tanaka's characterization of countable nonstandard models of WKL_0 using initial self-embeddings. In Section 3, we present a new proof of the conservativity of WKL_0 over RCA_0 for sentences of the form $\forall X\ \exists! Y\ \theta(X,Y)$, where θ is an arithmetical formula. This paper ends with a few open question.

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1. Background

We assume the reader is acquainted with rudiments of first- and second-order

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arithmetic [6, 7, 13]. Definability means parametric definability unless otherwise stated. We denote by $\mathcal{L}_{\mathrm{I}}$ and $\mathcal{L}_{\mathrm{II}}$ the usual languages for first- and second-order arithmetic respectively. There is *no* symbol for exponentiation in these languages. We only use lowercase letters to refer to numbers and uppercase letters to refer to sets. We write an $\mathcal{L}_{\mathrm{II}}$ structure as a pair $(M, \mathscr{X})$, where M is the universe for the first-order sort and $\mathscr{X}$ is the universe for the second-order sort. All elements of such an $\mathscr{X}$ are assumed to be subsets of M in view of our implicit inclusion of extensionality in all $\mathcal{L}_{\mathrm{II}}$ theories. If $\mathscr{X} = \{A\}$, then we often write $(M, \mathscr{X})$ as (M, A). We consider $\mathcal{L}_{\mathrm{II}}$ as an extension of $\mathcal{L}_{\mathrm{I}}$. In particular, the first-order part M of every $\mathcal{L}_{\mathrm{II}}$ structure $(M, \mathscr{X})$ is an $\mathcal{L}_{\mathrm{I}}$ structure itself. Recall that an *arithmetical* formula is an $\mathcal{L}_{\mathrm{II}}$ formula in which all quantifiers range over the first-order sort. While formulas in $\Sigma^0_0, \Delta^0_1, \Sigma^0_1, \ldots$ may involve second-order entities, those in $\Delta_0, \Sigma_1, \ldots$ may not.

All theories include $\mathrm{I}\Delta_0$. While $\mathrm{I}\Sigma_1 \vdash \mathrm{B}\Sigma_1$, the totality of $x \mapsto 2^x$, which we denote by exp, is provable in $\mathrm{I}\Sigma_1$ but not in $\mathrm{B}\Sigma_1$. The $\mathcal{L}_{\mathrm{II}}$ theory RCA^*_0 is axiomatized by $\mathrm{I}\Sigma^0_0 + \exp$ plus Δ^0_1 comprehension, and $\mathrm{RCA}_0 = \mathrm{RCA}^*_0 + \mathrm{I}\Sigma^0_1$. The *Weak König Lemma* (WKL) is an $\mathcal{L}_{\mathrm{II}}$ sentence which expresses 'every unbounded 0–1 tree has an unbounded branch' over RCA^*_0. Define $\mathrm{WKL}^*_0 = \mathrm{RCA}^*_0 + \mathrm{WKL}$ and $\mathrm{WKL}_0 = \mathrm{RCA}_0 + \mathrm{WKL}$. It is known [13, Section X.4] that $\mathrm{RCA}^*_0 \vdash \mathrm{B}\Sigma^0_1$.

The choice of the coding apparatus is immaterial, as long as its graph is Δ_0 and its basic properties are provable in $\mathrm{I}\Delta_0$. For definiteness, we choose the Ackermann coding using binary expansions of numbers [6, Section V.3].

Definition. If $M, K \models \mathrm{I}\Delta_0$, then write $K \supseteq_{\mathrm{e}} M$ to mean K is an end extension of M. Let $i \in \mathrm{Ack}(x)$ be a Δ_0 formula which expresses 'the ith digit in the binary expansion of x is 1' over $\mathrm{I}\Delta_0$. Suppose $M \subseteq_{\mathrm{e}} K \models \mathrm{I}\Delta_0$. For $c \in K$, define

$$\mathrm{Ack}(c/M) = \{i \in M : K \models i \in \mathrm{Ack}(c)\}.$$

Let $\mathrm{Cod}(K/M) = \{\mathrm{Ack}(c/M) : c \in K\}$.

The theorem below tells us that, roughly speaking, building models of WKL^*_0 is the same as building proper end extensions satisfying $\mathrm{I}\Delta_0$. It provides an iteration-free method of constructing models of WKL^*_0, unlike the usual constructions which involve iteratively adding branches through trees; cf. Section IX.2 in Simpson [13], for example. The ideas can be traced back to Scott's work [11] in the 1960s.

Theorem 1.1 (Enayat–Wong [3]). The following are equivalent for a countable $(M, \mathscr{X}) \models \mathrm{I}\Sigma^0_0 + \exp$.

(i) $(M, \mathscr{X}) \models \mathrm{WKL}^*_0$.
(ii) $\mathscr{X} = \mathrm{Cod}(K/M)$ for some $K \supsetneq_{\mathrm{e}} M$ satisfying $\mathrm{I}\Delta_0$.

Proof sketch. The proof of (ii) $\Rightarrow$ (i) consists of a standard absoluteness and overspill argument; see Exercise 13.1 in Kaye [7], for example. Our argument for (i) $\Rightarrow$ (ii) comes from the Arithmetized Completeness Theorem [7, Chapter 13].

First, notice recursive syntactical objects, such as $\mathcal{L}_\mathrm{I}$ and $\mathrm{I}\Delta_0$, have natural formalizations within WKL_0^*. Although WKL_0^* does not prove the consistency of $\mathrm{I}\Delta_0$, it does prove the *cut-free* consistency of $\mathrm{I}\Delta_0$, i.e., that there is no cut-free proof of $\bot$ from $\mathrm{I}\Delta_0$; see Corollary V.5.29 in Hájek–Pudlák [6] and Theorem 7.2.3 in Visser [16].

Fix a countable $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$. Let C be an unbounded subset of M in $\mathscr{X}$, and let $\mathcal{L}$ be the language in $(M, \mathscr{X})$ obtained from $\mathcal{L}_\mathrm{I}$ by adding a new constant symbol c for each $c \in C$. We build by recursion an ω-sequence

$$\mathrm{I}\Delta_0 = T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$$

of elements of $\mathscr{T}$, where $\mathscr{T}$ denotes the set of all cut-free consistent $\mathcal{L}$ theories in $(M, \mathscr{X})$ in which only boundedly many $c \in C$ appear.

Use the T_{3n+1}'s to ensure $\mathrm{Std}(T_\omega) = \mathbb{N} \cap \bigcup_{n\in\mathbb{N}} T_n$ is a complete, consistent, Henkinized $\mathcal{L}$ theory when viewed externally. Let K be the Henkin model of $\mathrm{Std}(T_\omega)$. We know $K \models \mathrm{I}\Delta_0$ because $\mathrm{Std}(T_\omega) \supseteq \mathrm{I}\Delta_0$. Every $m \in M$ is represented by a closed $\mathcal{L}$ term $\check{m}$ in $(M, \mathscr{X})$. Therefore, in the same way one views every model of $\mathrm{I}\Delta_0$ as an end extension of $\mathbb{N}$, we can view K as an end extension of M.

The T_{3n+2}'s are used to ensure $\mathrm{Cod}(K/M) \subseteq \mathscr{X}$. Using the countability of M, enumerate $C = \{c_n : n \in \mathbb{N}\}$. Given any $T_{3n+1} \in \mathscr{T}$, use WKL to find $T_{3n+2} \in \mathscr{T}$ extending T_{3n+1} which decides $\check{m} \in \mathrm{Ack}(c_n)$ for every $m \in M$. Then Δ_1^0 comprehension implies, for every $n \in \mathbb{N}$,

$$\mathrm{Ack}(c_n/M) = \{m \in M : T_{3n+2} \text{ contains the formula } \check{m} \in \mathrm{Ack}(c_n)\} \in \mathscr{X}.$$

Finally, we use the T_{3n+3}'s to ensure $\mathscr{X} \subseteq \mathrm{Cod}(K/M)$. Use the countability of $\mathscr{X}$ to enumerate $\mathscr{X} = \{A_n : n \in \mathbb{N}\}$. Given any $T_{3n+2} \in \mathscr{T}$, find $c \in C$ which does not appear in T_{3n+2}, and use it to code A_n by setting

$$T_{3n+3} = T_{3n+2} \cup \{\check{m} \in \mathrm{Ack}(c) : m \in A_n\} \cup \{\check{m} \notin \mathrm{Ack}(c) : m \in M \setminus A_n\}. \qquad \square$$

2. Initial self-embedding

Definition. An *initial self-embedding* of an $\mathcal{L}_\mathrm{I}$ structure M is an embedding $M \to M$ whose image is a proper initial segment of M.

H. Friedman [4] showed that every countable nonstandard model of PA has an initial self-embedding in the 1970s. Soon after, it was recognized that the full strength of PA is not needed for the construction of initial self-embeddings. In the 1980s, following a series of partial results by various researchers, Dimitracopoulos and Paris [1, Corollary 2.4] and independently Ressayre [10, Theorem 1.I] brought PA down to $\mathrm{I}\Sigma_1$. As Ressayre proved, this is optimal, in a sense.

Theorem 2.1 (Ressayre)**.** The following are equivalent for a countable $M \models \mathrm{I}\Delta_0$.

(i) $M \models \mathrm{I}\Sigma_1$ and $M \not\cong \mathbb{N}$.
(ii) For each $a \in M$, there is an initial self-embedding of M which fixes all $x < a$.

Proof. See the proof of Theorem 2.2 below for an argument that demonstrates (ii) $\Rightarrow$ (i). For (i) $\Rightarrow$ (ii), one can use a direct back-and-forth construction extracted from Kaye [7, Chapter 12] and Hájek–Pudlák [5, Appendix]. ☐

It was Tanaka [15] who introduced initial self-embeddings into the $\mathcal{L}_{\mathrm{II}}$ context.

Definition. An *initial self-embedding* j of an $\mathcal{L}_{\mathrm{II}}$ structure $(M, \mathscr{X})$ consists of an initial self-embedding j_1 of M and an injection $j_2 \colon \mathscr{X} \to \mathscr{X}$ which together preserve $\in$ and $\notin$, and satisfy the following 'surjectivity' condition:

$$\forall V{\in}\mathscr{X} \quad \exists U{\in}\mathscr{X} \qquad j_2(U) \cap \mathrm{Im}(j_1) = V \cap \mathrm{Im}(j_1). \tag{$*$}$$

We mention here several natural properties that follow from this definition. Let j be an initial self-embedding of $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$. Define $I = \mathrm{Im}(j_1)$ and $\mathscr{X}{\upharpoonright}I = \{X \cap I : X \in \mathscr{X}\}$. On the one hand, we see that $j = j_1 \cup j_2$ is really an embedding between $\mathcal{L}_{\mathrm{II}}$ structures in the usual model-theoretic sense. On the other hand, because of $(*)$, we can view j as an isomorphism $(M, \mathscr{X}) \to (I, \mathscr{X}{\upharpoonright}I)$ which sends each $S \in \mathscr{X}$ to $j_2(S) \cap I$. Notice $\mathscr{X}{\upharpoonright}I = \mathrm{Cod}(M/I)$ since $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$.

Theorem 2.2 (Tanaka)**.** The following are equivalent for a countable $(M, \mathscr{X}) \models \mathrm{I}\Sigma_0^0$.

(i) $(M, \mathscr{X}) \models \mathrm{WKL}_0$ and $M \not\cong \mathbb{N}$.
(ii) For each $a \in M$ and each $S \in \mathscr{X}$, there is an initial self-embedding of $(M, \mathscr{X})$ which fixes S and all $x < a$.

Proof sketch. First, consider (i) $\Rightarrow$ (ii). Tanaka's original proof [15, Section 2] is a refinement of Ressayre's argument for Theorem 2.1. Here we describe an alternative proof from Enayat [2]. Use the nonstandardness of M and $\mathrm{I}\Sigma_1^0$ in the form of strong Σ_1^0 collection [6, Theorem I.2.23(1)] to find $b \in M$ and $I \subseteq_{\mathrm{e}} M$ closed under $x \mapsto 2^x$ containing a such that for every $x \in I$ and every Π_0^0 formula $\xi(x, y, Z)$,

$$(M, \mathscr{X}) \models \exists y\ \xi(x, y, S) \to \exists y{<}b\ \xi(x, y, S). \tag{$\dagger$}$$

Next, build a proper end extension (K, S^*) of (M, S) satisfying $\mathrm{I}\Sigma_0^0$ in which $\mathrm{Cod}(K/M) = \mathscr{X}$ and our condition on b and I remains true. (Here, saying (K, S^*) is an *end extension* of (M, S) means simply $K \supseteq_{\mathrm{e}} M$ and $S^* \subseteq K$ with $S^* \cap M = S$.) This can be achieved by running our proof of (i) $\Rightarrow$ (ii) for Theorem 1.1 with $\mathcal{L}$

redefined to be $\mathcal{L}_{\mathrm{II}} \cup C$ and

$$T_0 \supseteq \mathrm{I}\Sigma^0_0 \cup \{\exists! Z\ (Z = Z)\} \cup \{\forall Z\ (\check{m} \in Z) : m \in S\} \cup \{\forall Z\ (\check{m} \notin Z) : m \in M \setminus S\} \\ \cup \{\forall Z\ \forall y\ \neg\xi(\check{x}, y, Z) : \xi \in \Pi^0_0 \text{ and } x < b \text{ where } (M, \mathscr{X}) \models \forall y\ \neg\xi(x, y, S)\}.$$

As $(M, S) \models \mathrm{I}\Sigma^0_1$, one can use an indicator argument [8, page 252] to shorten (K, S^*) to a proper end extension of (M, S) which satisfies $\mathrm{I}\Sigma^0_1$. Therefore, without loss of generality, we assume $(K, S^*) \models \mathrm{I}\Sigma^0_1$. Apply a straightforward generalization of Theorem 2.1 to obtain an initial self-embedding j of (K, S^*) which fixes every $x \in I$ and whose image lies below b. Then $j_1\restriction M$ induces an embedding we want.

Now, consider (ii) ⇒ (i). Theorem 1.1 already tells us $(M, \mathscr{X}) \models \mathrm{WKL}^*_0$. So it suffices to show that $(M, \mathscr{X})$ satisfies the least number principle for Σ^0_1 formulas [6, Theorem I.2.4]. Consider the Σ^0_1 formula $\exists y\ \xi(x, y, Z)$ where $\xi \in \Pi^0_0$. Let $a \in M$ and $S \in \mathscr{X}$. Apply (ii) to obtain an initial self-embedding j of $(M, \mathscr{X})$ which fixes S and all $x < a$. Define $I = \mathrm{Im}(j_1)$ and pick any $b \in M \setminus I$. We claim that

$$\{x < a : (M, \mathscr{X}) \models \exists y\ \xi(x, y, S)\} = \{x < a : (M, \mathscr{X}) \models \exists y{<}b\ \xi(x, y, S)\}.$$

This will give what we want because the set on the right-hand side must have a least element by $\mathrm{I}\Sigma^0_0$ if it is nonempty. For the non-trivial direction, suppose $x < a$ such that $(M, \mathscr{X}) \models \exists y\ \xi(x, y, S)$. Then $(I, \mathscr{X}\restriction I) \models \exists y\ \xi(j_1(x), y, j_2(S) \cap I)$, viewing j as an isomorphism $(M, \mathscr{X}) \to (I, \mathscr{X}\restriction I)$. As j fixes S and every element less than a, one sees actually $(I, \mathscr{X}\restriction I) \models \exists y\ \xi(x, y, S \cap I)$. Since $b > I \subseteq_{\mathrm{e}} M$ and $\xi \in \Pi^0_0$, we deduce that $(M, \mathscr{X}) \models \exists y{<}b\ \xi(x, y, S)$, as required. □

If we do not require the initial self-embeddings to fix arbitrarily large proper initial segments, then $\mathrm{I}\Sigma_1$ can be further reduced to $\mathrm{B}\Sigma_1 + \exp$ plus some recursive saturation, as the next theorem from Dimitracopoulos–Paris [1, Theorem 2.2] shows. Let $\mathcal{L} \in \{\mathcal{L}_{\mathrm{I}}, \mathcal{L}_{\mathrm{II}}\}$ and Γ be a class of $\mathcal{L}$ formulas. Then we write $\Delta_0(\Gamma)$ for the closure of Γ under Boolean operations and bounded quantification. As is well known [6, Lemma I.2.14], if $I \subsetneq_{\mathrm{e}} M \models \mathrm{I}\Sigma_1$, then I cannot be $\Delta_0(\Sigma_1)$-definable in M. A subset of an $\mathcal{L}$ structure is said to be $\Delta^-_0(\Gamma)$-*definable over certain parameters* if this subset is definable by a $\Delta_0(\Gamma)$ formula in which the parameters only appear in the scope of an unbounded quantifier.

Theorem 2.3 (Dimitracopoulos–Paris)**.** The following are equivalent for all countable $M \models \mathrm{I}\Delta_0 + \exp$ and all $a \in M$.

(i) $M \models \mathrm{B}\Sigma_1$ and $\mathbb{N}$ is not $\Delta^-_0(\Sigma_1)$-definable in M over the parameter a.
(ii) There is an initial self-embedding of M which fixes a.

Proof. See the proof of Theorem 2.4 below for some details. The self-embedding in (ii) can be built using a direct back-and-forth construction from Kaye [7, Chapter 12]. □

In particular, a countable $M \models \mathrm{B}\Sigma_1 + \exp$ has an initial self-embedding if and only if $\mathbb{N}$ is not parameter-free $\Delta_0(\Sigma_1)$-definable in M. Similarly, the $\mathcal{L}_{\mathrm{II}}$ version below from Enayat–Wong [3] implies that a countable $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$ has an initial self-embedding if and only if $\mathbb{N}$ is not parameter-free $\Delta_0(\Sigma_1)$-definable in M.

Theorem 2.4 (Enayat–Wong)**.** Fix a countable $(M, \mathscr{X}) \models \mathrm{I}\Sigma_0^0 + \exp$. Let $a \in M$ and $S \in \mathscr{X}$. The following are equivalent.

(i) $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$ and $\mathbb{N}$ is not $\Delta_0^-(\Sigma_1^0)$-definable in $(M, \mathscr{X})$ over a and S.
(ii) There is an initial self-embedding of $(M, \mathscr{X})$ which fixes a and S.

Proof sketch. The proof of (i) $\Rightarrow$ (ii) is similar to that in Theorem 2.2, using Theorem 2.3 in place of Theorem 2.1. So we only indicate here the main differences. First, we clearly cannot ensure $(K, S^*) \models \mathrm{I}\Sigma_1^0$, but finding a recursively saturated $(K, S^*) \models \mathrm{B}\Sigma_1^0$ as in Kaye [7, Section 14.2] is enough for the application of Theorem 2.3; see also the proof of (iii) $\Rightarrow$ (ii) in Dimitracopoulos–Paris [1, Theorem 2.2]. Another tricky point is in obtaining $b \in M$ which makes (†) hold for $x = a$ and every $\xi \in \Pi_0^0$. We describe here the argument from Dimitracopoulos–Paris [1, Theorem 2.2]. As $\mathbb{N}$ is not $\Delta_0^-(\Sigma_1^0)$-definable in $(M, \mathscr{X})$ over a and S, overspill gives us $\nu \in M \setminus \mathbb{N}$ such that $(M, \mathscr{X}) \models \beta(\nu, a, S)$, where $\beta(n, a, S)$ is the $\Delta_0(\Sigma_1^0)$ formula

$$\exists \zeta{<}n \ \Big(\exists v \ \Pi_0^S\text{-Sat}(\zeta, [a, v]) \wedge \forall v \ \Big(\Pi_0^S\text{-Sat}(\zeta, [a, v]) \\ \rightarrow \forall \xi{<}n \ \big(\exists y \ \Pi_0^S\text{-Sat}(\xi, [a, y]) \rightarrow \exists y{\leqslant}v \ \Pi_0^S\text{-Sat}(\xi, [a, y])\big)\Big)\Big)$$

and $\Pi_0^S\text{-Sat}(\zeta, [x, y])$ is a Σ_1^0 predicate asserting the satisfaction of the Π_0^0 formula $\zeta(x, y, S)$. Let ζ be a witness to the satisfaction of $\beta(\nu, a, S)$ in $(M, \mathscr{X})$. Then any $b \in M$ such that $(M, \mathscr{X}) \models \Pi_0^S\text{-Sat}(\zeta, [a, b])$ does the job.

The argument for (ii) $\Rightarrow$ (i) also comes from Dimitracopoulos–Paris [1, Theorem 2.2]. Fix an initial self-embedding j of $(M, \mathscr{X})$ which fixes a and S. Suppose $(M, \mathscr{X}) \models \theta(n, a, S)$ for all $n \in \mathbb{N}$, where $\theta(w, y, Z)$ is the $\Delta_0(\Sigma_1^0)$ formula

$$\exists x_1{<}t_1(w) \ \forall x_2{<}t_2(w, x_1) \cdots \bigwedge_{i<k} \big(\exists u \ \xi_i(u, w, \bar{x}, y, Z) \vee \forall v \ \zeta_i(v, w, \bar{x}, y, Z)\big)$$

and the ξ_i's and ζ_i's are all Σ_0^0. Let $I = \mathrm{Im}(j_1)$. Take any $b \in M \setminus I$, and set $d = j_1(b)$. Then, as in our proof of Theorem 2.2, we can show for all $n, \bar{x} \in \mathbb{N}$ and all $i < k$,

$$(M, \mathscr{X}) \models \exists u \ \xi_i(u, n, \bar{x}, a, S) \vee \forall v \ \zeta_i(v, n, \bar{x}, a, S) \\ \leftrightarrow \exists u{<}d \ \xi_i(u, n, \bar{x}, a, S) \vee \forall v{<}b \ \zeta_i(v, n, \bar{x}, a, S).$$

Therefore, an application of Σ_0^0 overspill gives us $\nu \in I \setminus \mathbb{N}$ such that $(M, \mathscr{X})$ satisfies

$$\exists x_1{<}t_1(\nu) \ \forall x_2{<}t_2(\nu, x_1) \cdots \bigwedge_{i<k} \big(\exists u{<}d \ \xi_i(u, \nu, \bar{x}, a, S) \vee \forall v{<}b \ \zeta_i(v, \nu, \bar{x}, a, S)\big).$$

As $d \in I < b$, this easily implies $(I, \mathscr{X}{\upharpoonright}I) \models \theta(\nu, a, S \cap I)$. Recall we can view j as an isomorphism $(M, \mathscr{X}) \to (I, \mathscr{X}{\upharpoonright}I)$. Since j fixes a and S, it follows that $(M, \mathscr{X}) \models \theta(j_1^{-1}(\nu), a, S)$ where $j_1^{-1}(\nu) > \mathbb{N}$, as required. □

3. Set extension

Definition. A *set extension* of an $\mathcal{L}_{\mathrm{II}}$ structure is an extension that adds only sets.

Harrington [13, Section IX.2] proved that WKL_0 is Π^1_1-conservative over RCA_0 using set extensions. We know WKL_0 is not Σ^1_1-conservative over RCA_0 because WKL_0 proves the existence of a consistent completion of $\mathrm{I}\Delta_0$ [13, Theorem II.8.11 and Theorem IV.3.3], but RCA_0 does not by the Incompleteness Theorem. Tanaka conjectured in 1995 that the conservativity between WKL_0 and RCA_0 holds for the larger class of formulas specified below. His conjecture was affirmed by Simpson, Tanaka, and Yamazaki [14, Theorem 4.18] in 2002.

Theorem 3.1 (Simpson–Tanaka–Yamazaki)**.** If $\mathrm{WKL}_0 \vdash \forall X\ \exists! Y\ \theta(X, Y)$ where $\theta(X, Y)$ is an arithmetical formula, then $\mathrm{RCA}_0 \vdash \forall X\ \exists! Y\ \theta(X, Y)$.

Simpson, Tanaka, and Yamazaki [14] proved their conservation theorem using set extensions too, via the following theorem. If $(M, \mathscr{X})$ is an $\mathcal{L}_{\mathrm{II}}$ structure, then we denote by $\Delta^0_1\text{-Def}(M, \mathscr{X})$ the collection of all Δ^0_1-definable subsets of M in $(M, \mathscr{X})$. A model $(M, \mathscr{X}) \models \mathrm{RCA}^*_0$ is *principal* if $\mathscr{X} = \Delta^0_1\text{-Def}(M, A)$ for some $A \in \mathscr{X}$.

Theorem 3.2 (Simpson–Tanaka–Yamazaki)**.** Every countable principal $(M, \mathscr{X}) \models \mathrm{RCA}_0$ has set extensions $(M, \mathscr{Y}_1), (M, \mathscr{Y}_2) \models \mathrm{WKL}_0$ such that

(a) $\mathscr{Y}_1 \cap \mathscr{Y}_2 = \mathscr{X}$; and
(b) if $\eta(\bar{v}, \bar{X})$ is an $\mathcal{L}_{\mathrm{II}}$ formula, then for all $\bar{m} \in M$ and all $\bar{S} \in \mathscr{X}$,

$$(M, \mathscr{Y}_1) \models \eta(\bar{m}, \bar{S}) \quad \Leftrightarrow \quad (M, \mathscr{Y}_2) \models \eta(\bar{m}, \bar{S}).$$

Proof of Theorem 3.1 from Theorem 3.2. Suppose $\mathrm{WKL}_0 \vdash \forall X\ \exists! Y\ \theta(X, Y)$, where θ is an arithmetical formula. Take any countable $(M, \mathscr{X}_0) \models \mathrm{RCA}_0$ and any $A \in \mathscr{X}_0$. Set $\mathscr{X} = \Delta^0_1\text{-Def}(M, A)$. Apply Theorem 3.2 to find $(M, \mathscr{Y}_1), (M, \mathscr{Y}_2) \models \mathrm{WKL}_0$ satisfying (a) and (b) there. Using the choice of θ, locate the unique $B_1 \in \mathscr{Y}_1$ such that $(M, \mathscr{Y}_1) \models \theta(A, B_1)$ and the unique $B_2 \in \mathscr{Y}_2$ such that $(M, \mathscr{Y}_2) \models \theta(A, B_2)$. By (b),

$$(M, \mathscr{Y}_1) \models \forall Y\ \big(\theta(A, Y) \to m \in Y\big) \quad \Leftrightarrow \quad (M, \mathscr{Y}_2) \models \forall Y\ \big(\theta(A, Y) \to m \in Y\big)$$

for every $m \in M$. So $B_1 = B_2$ by extensionality. The set B_1, being equal to B_2, is in both $\mathscr{Y}_1$ and $\mathscr{Y}_2$. As a result, it must be in $\mathscr{X}$ by (a), and hence in $\mathscr{X}_0$ by Δ^0_1 comprehension. Thus $(M, \mathscr{X}_0) \models \exists Y\ \theta(A, Y)$. Such a witness Y must be unique because

$(M, \mathscr{X}_0)$ has a set extension $(M, \mathscr{Y}_0) \models \mathrm{WKL}_0$ and $\mathrm{WKL}_0 \vdash \forall X\ \exists! Y\ \theta(X, Y)$. □

The original proof of Theorem 3.2 involves forcing with universal trees [14, Sections 3 and 4]. We sketch below a new proof inspired by our construction for Theorem 1.1. The details will be spelt out in a sequel to Enayat–Wong [3]. The idea of using the Pour-El–Kripke paper [9] comes from the original proof. See Simpson [12] for more arguments with a similar flavour.

Proof sketch of Theorem 3.2. Take a countable $(M, \mathscr{X}) \models \mathrm{RCA}_0$ and $A \in \mathscr{X}$ such that $\mathscr{X} = \Delta^0_1\text{-Def}(M, A)$. Define $C, \mathcal{L}, \check{m}, \ldots$ as in the proof of Theorem 1.1. Fix a distinguished $a \in C$. Set

$$\mathrm{I}\Delta_0^A = \mathrm{I}\Delta_0 \cup \{\check{m} \in \mathrm{Ack}(a) : m \in A\} \cup \{\check{m} \notin \mathrm{Ack}(a) : m \in M \setminus A\}.$$

Let $\hat{\mathscr{T}}$ be the collection of all consistent $\mathcal{L}$ theories $T \supseteq \mathrm{I}\Delta_0^A$ in $(M, \mathscr{X})$ such that

(1) for every $\mathcal{L}$ formula $\theta(\bar{v})$ in $(M, \mathscr{X})$, there exist $\bar{c} \in C$ such that T contains the $\mathcal{L}$ sentence $\exists \bar{v}\ \theta(\bar{v}) \rightarrow \theta(\bar{c})$; and
(2) for every sequence $(\zeta_m)_{m \in M}$ of $\mathcal{L}$ sentences in $(M, \Delta^0_1\text{-Def}(M, A))$ involving only boundedly many constant symbols from C, there exists $c \in C$ such that T contains, for every $m \in M$, the sentence $\zeta_m \leftrightarrow \check{m} \in \mathrm{Ack}(c)$.

The forcing poset is $\hat{\mathbb{T}} = (\hat{\mathscr{T}}, \supseteq)$. We follow standard forcing terminology as in Simpson–Tanaka–Yamazaki [14]. Given an expansion $\mathfrak{M}$ of $(M, \mathscr{X})$, say a filter in $\hat{\mathbb{T}}$ is $\mathfrak{M}$-*generic* if it meets all $\mathfrak{M}$-definable dense sets of conditions.

Let $\mathscr{G}$ be an $(M, \mathscr{X})$-generic filter. For each $c \in C$, define

$$\mathrm{Set}(c/\mathscr{G}) = \big\{m \in M : \text{the formula } \check{m} \in \mathrm{Ack}(c) \text{ is in } \bigcup \mathscr{G}\big\}.$$

Set $\mathscr{X} \wr \mathscr{G} = \{\mathrm{Set}(c/\mathscr{G}) : c \in C\}$. By considering the Henkin model of $\mathrm{Std}(\mathscr{G}) = \mathbb{N} \cap \bigcup \mathscr{G}$, we know $\mathscr{X} \wr \mathscr{G} = \mathrm{Cod}(K/M)$ for some $K \supsetneq_{\mathrm{e}} M$ satisfying $\mathrm{I}\Delta_0$. So Theorem 1.1 implies $(M, \mathscr{X} \wr \mathscr{G}) \models \mathrm{WKL}_0^*$. It follows that $\mathscr{X} = \Delta^0_1\text{-Def}(M, A) \subseteq \mathscr{X} \wr \mathscr{G}$ because $A = \mathrm{Set}(a/\mathscr{G}) \in \mathscr{X} \wr \mathscr{G}$ by construction. As

$$\big\{T \in \hat{\mathscr{T}} : (M, \mathscr{X}) \models \forall m{<}b\ \big(T \supseteq \{\zeta(\check{m}, \check{v}) : v \in M\} \vee \exists d\ T \supseteq \{\exists v{\leqslant}\check{d}\ \neg\zeta(\check{m}, v)\}\big)\big\}$$

is dense in $\hat{\mathbb{T}}$ for each $b \in M$ and each $\mathcal{L}$ formula ζ, one can show $(M, \mathscr{X} \wr \mathscr{G}) \models \mathrm{I}\Sigma^0_1$.

The *forcing language* $\mathcal{L}_{\mathrm{II}}^{\mathrm{F}}$ has, in addition to the symbols in $\mathcal{L}_{\mathrm{II}}$, a constant symbol m of the first-order sort for each $m \in M$, and a constant symbol $\mathrm{Set}(c)$ of the second-order sort for each $c \in C$. When writing an $\mathcal{L}_{\mathrm{II}}^{\mathrm{F}}$ formula, we *always* display these new constant symbols. If $T \in \hat{\mathscr{T}}$ and $\eta(\bar{m}, \mathrm{Set}(\bar{c}))$ is an $\mathcal{L}_{\mathrm{II}}^{\mathrm{F}}$ sentence, then write $T \Vdash \eta(\bar{m}, \mathrm{Set}(\bar{c}))$ to mean $(M, \mathscr{X} \wr \mathscr{G}) \models \eta(\bar{m}, \mathrm{Set}(\bar{c}/\mathscr{G}))$ for all $(M, \mathscr{X})$-generic filters $\mathscr{G}$ in $\hat{\mathbb{T}}$ containing T. As usual, one can prove the *Truth Lemma* in this context:

for all $(M,\mathscr{X})$-generic filters $\mathscr{G}$ and all $\mathcal{L}_{\mathrm{II}}^{\mathrm{F}}$ sentences $\eta(\bar{m},\mathrm{Set}(\bar{c}))$,

$$(M,\mathscr{X}\wr\mathscr{G})\models\eta(\bar{m},\mathrm{Set}(\bar{c}/\mathscr{G}))\quad\Leftrightarrow\quad\exists T{\in}\mathscr{G}\ T\Vdash\eta(\bar{m},\mathrm{Set}(\bar{c})).$$

Pour-El and Kripke [9] proved that the Lindenbaum algebras of any two recursively axiomatized 'effectively inseparable' theories are recursively isomorphic. Their proof can be formalized directly in $\mathrm{I}\Sigma_1^0$. This implies the following homogeneity property of $\hat{\mathbb{T}}$: given any $T_1,T_2\in\hat{\mathscr{T}}$, there is a bijection $F\colon\mathrm{Snt}(\mathcal{L})\to\mathrm{Snt}(\mathcal{L})$ in $\mathscr{X}$ preserving the Boolean operations such that

$$(M,\mathscr{X})\models\forall\theta{\in}\mathrm{Snt}(\mathcal{L})\ \big(\mathrm{Con}(T_1\cup\{\theta\})\leftrightarrow\mathrm{Con}(T_2\cup\{F(\theta)\})\big).$$

Here $\mathrm{Snt}(\mathcal{L})$ denotes the set of all $\mathcal{L}$ sentences in $(M,\mathscr{X})$. One can additionally require that, with the possible exception of a, the constant symbols from C involved in θ and $F(\theta)$ are always the same. Such an F will be called an *isomorphism* from T_1 to T_2. The hypothesis $\mathscr{X}=\Delta_1^0\text{-}\mathrm{Def}(M,A)$ is invoked here.

We are now ready for the actual proof. Take any $(M,\mathscr{X})$-generic filter $\mathscr{G}_1$ in $\hat{\mathbb{T}}$. Let $\mathscr{G}_2$ be an $(M,\mathscr{X},\mathscr{G}_1)$-generic filter in $\hat{\mathbb{T}}$. Set $\mathscr{Y}_1=\mathscr{X}\wr\mathscr{G}_1$ and $\mathscr{Y}_2=\mathscr{X}\wr\mathscr{G}_2$. We already saw that $(M,\mathscr{Y}_1),(M,\mathscr{Y}_2)\models\mathrm{WKL}_0$. As $\mathscr{G}_2$ is $(M,\mathscr{X},\mathscr{G}_1)$-generic, one can show $\mathscr{Y}_1\cap\mathscr{Y}_2=\mathscr{X}$ in a way similar to how one avoids coding a particular non-recursive $X\subseteq\mathbb{N}$ when building a nonstandard model of PA [7, Lemma 11.2].

It remains to establish (b). Fix $\bar{m}\in M$ and $\bar{S}\in\mathscr{X}$. Consider the $\mathcal{L}_{\mathrm{II}}$ formula $\eta(\bar{v},\bar{X})$. Suppose $(M,\mathscr{Y}_1)\models\eta(\bar{m},\bar{S})$. Pick any $T_1\in\mathscr{G}_1$. Find $\bar{c}\in C$ which make $\mathrm{Set}(\bar{c}/\mathscr{G}_2)=\bar{S}$. Apply the Truth Lemma to find a sufficiently large $T_2\in\mathscr{G}_2$ such that $T_2\Vdash\eta(\bar{m},\mathrm{Set}(\bar{c}))$ or $T_2\Vdash\neg\eta(\bar{m},\mathrm{Set}(\bar{c}))$. The homogeneity of $\hat{\mathbb{T}}$ then gives an isomorphism F from T_1 to T_2 in $\mathscr{X}$. Via F, the $(M,\mathscr{X})$-generic filter $\mathscr{G}_1$ is mapped to an $(M,\mathscr{X})$-generic filter $\mathscr{G}_1'$ containing T_2 because $F\in\mathscr{X}$. Moreover, our additional requirement on the isomorphism F, when coupled with condition (2) on $\hat{\mathscr{T}}$, implies $\mathscr{X}\wr\mathscr{G}_1=\mathscr{X}\wr\mathscr{G}_1'$. Thus $(M,\mathscr{X}\wr\mathscr{G}_1')\models\eta(\bar{m},\bar{S})$ and so $T_2\nVdash\neg\eta(\bar{m},\mathrm{Set}(\bar{c}))$. It follows that $T_2\Vdash\eta(\bar{m},\mathrm{Set}(\bar{c}))$. Hence $(M,\mathscr{Y}_2)\models\eta(\bar{m},\bar{S})$. □

As the reader may already have noticed, our constructions for Theorem 1.1 and Theorem 3.2 are really the same construction executed in two different ways.

Using a separate forcing construction, Simpson, Tanaka, and Yamazaki [14, Corollary 5.16] managed to eliminate the principality assumption in Theorem 3.2.

4. Questions

1. Given $(M,\mathscr{X})\models\mathrm{WKL}_0$, can one always find $K\supsetneq_{\mathrm{e}}M$ satisfying $\mathrm{I}\Delta_0$ such that $\mathrm{Cod}(K/M)=\mathscr{X}$? (The special case when $M=\mathbb{N}$ is known as Scott's Problem.)
2. Does every $(M,\mathscr{X})\models\mathrm{RCA}_0^*$ have a set extension $(M,\mathscr{Y})\models\mathrm{WKL}_0^*$?

3. Does every countable $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$ have set extensions $(M, \mathscr{Y}_1), (M, \mathscr{Y}_2) \models \mathrm{WKL}_0^*$ such that (a) and (b) in Theorem 3.2 hold?

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