Upgrading the Arithmetized Completeness Theorem

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Upgrading the Arithmetized Completeness Theorem

Conservation Theorem (Simpson–Tanaka–Yamazaki 2002)

For all arithmetical formulas θ,

$$\mathsf{WKL}_0 \vdash \forall X\ \exists! Y\ \theta(X, Y) \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \forall X\ \exists! Y\ \theta(X, Y).$$

Plan of the talk

1. Introduction
2. Basic construction — Arithmetized Completeness Theorem
3. Upgrade — $\times$ forcing
4. Further application

Background

Fundamental Theorem of Algebra

Conservation Theorem (Simpson–Tanaka–Yamazaki 2002)

For all arithmetical formulas θ,

$$\underbrace{\mathsf{WKL}_0}_{\mathsf{RCA}_0 + \mathsf{WKL}} \vdash \forall X\ \exists! Y\ \theta(X, Y) \quad \Rightarrow \quad \underbrace{\mathsf{RCA}_0}_{\mathsf{PA}^- + \text{extensionality} + \mathsf{I}\Sigma^0_1 + \Delta^0_1\text{-comprehension}} \vdash \forall X\ \exists! Y\ \theta(X, Y).$$

Weak König Lemma (WKL)

Every unbounded 0–1 tree has an unbounded branch.

Conservativity of WKL over RCA_0

Harrington 1977, Ratajczyk 1983:	yes	for $\forall X\ \theta(X)$	θ arithmetical
Gödel 1931:	no	for $\exists X\ \theta(X)$	
Tanaka 1995:	??	for $\forall X\ \exists! Y\ \theta(X, Y)$	
Fernandes 2002:	yes	for $\forall X\ \exists! Y\ \eta(X, Y)$	$\eta \in \Sigma^0_3$

Model theory

Conservation Theorem (Simpson–Tanaka–Yamazaki 2002)

For all arithmetical formulas θ,

$$\mathsf{WKL}_0 \vdash \forall X\ \exists! Y\ \theta(X, Y) \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \forall X\ \exists! Y\ \theta(X, Y).$$

Main Lemma (Simpson–Tanaka–Yamazaki 2002)

Every countable $(M, \mathscr{X}) \models \mathsf{RCA}_0$ extends to $(M, \mathscr{Y}), (M, \mathscr{Y}') \models \mathsf{WKL}_0$ such that

(A) $\mathscr{Y} \cap \mathscr{Y}' = \mathscr{X}$; and

(B) for every $\mathscr{L}_{\mathrm{II}}(M, \mathscr{X})$-sentence σ,

$$(M, \mathscr{Y}) \models \sigma \quad \Leftrightarrow \quad (M, \mathscr{Y}') \models \sigma.$$

may mention parameters from $(M, \mathscr{X})$

$\sigma \quad = \quad$ "m is in the unique Y such that $\theta(A, Y)$"

Subsets coded in an end extension

Definition

Let $M \subseteq_{\mathrm{e}} K \models \mathrm{I}\Delta_0$. Then

$$\mathrm{Cod}(K/M) = \Big\{\overbrace{\{m \in M : \text{the } m\text{th prime divides } c\}}^{c\ \textit{codes}\ \text{this set}} : c \in K\Big\}.$$

iteration-free!

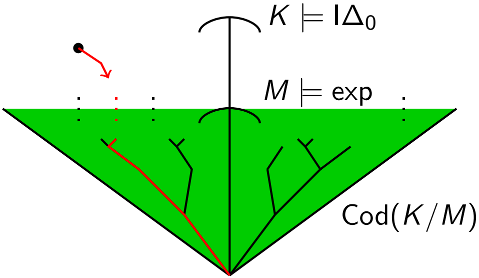
Theorem (Scott 1962?, Ratajczyk 1983)

If $M \models \mathrm{I}\Delta_0 + \exp$ and $K \models \mathrm{I}\Delta_0$ properly end-extending M, then $(M, \mathrm{Cod}(K/M)) \models \mathrm{WKL}_0^* \supseteq \mathrm{I}\Sigma_0^0 + \exp + \Delta_1^0\text{-comprehension} + \mathrm{WKL}$.

Proof

Overspill. $\square$

expansion $(M, \mathscr{X}) \models \mathrm{WKL}$
$\sim$ end extension $K \models \mathrm{I}\Delta_0$



Arithmetized Completeness Theorem

$(M, \mathscr{Y}) \models \mathsf{WKL}_0 - \mathsf{I}\Sigma^0_1$
by Scott and Ratajczyk

Global assumption

Fix a countable $(M, \mathscr{X}) \models \mathsf{RCA}_0$ and $A \in \mathscr{X}$ such that

$$\mathscr{X} = \Delta^0_1\text{-Def}(M, A).$$

Construction

Start with $T_0 = \mathsf{I}\Delta^A_0 = \mathsf{I}\Delta_0 +$ "$\dot{a}$ codes A" $+$ Henkin axioms. Build

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$$

with $T_n \in \mathscr{X}$ and $(M, \mathscr{X}) \models \mathrm{Con}(T_n)$ for all $n < \omega$.

- K is the Henkin model of the set of all formulas of standard shapes in $T_\omega = \bigcup_{n<\omega} T_n$.
- $K \supseteq_{\mathrm{e}} M$ because provably in RCA_0,

$$\mathsf{I}\Delta_0 \vdash \forall x < \check{m} \bigvee_{k<m} x = \check{k} \qquad \text{for all } m.$$

- $\mathscr{Y} = \mathrm{Cod}(K/M) \supseteq \Delta^0_1\text{-Def}(M, A) = \mathscr{X}$.

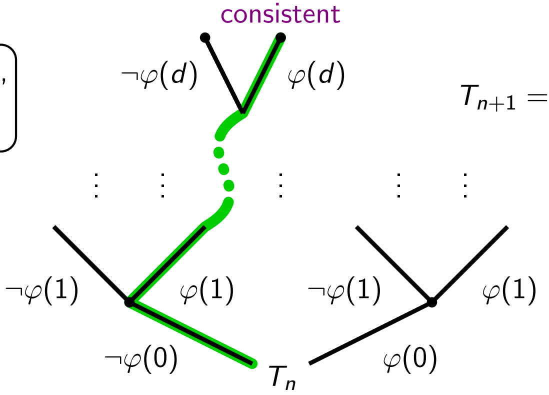
Preserving Σ^0_1-induction

Theorem ($\sim$ Harrington 1977)

We can make $(M, \mathscr{Y}) \models \mathsf{I}\Sigma^0_1$.

Proof

$$\boxed{\begin{array}{c} \varphi(x) \in \Sigma^0_1, \\ d \in M \end{array}}$$



$$T_{n+1} = T_n + \text{path}$$

$\square$

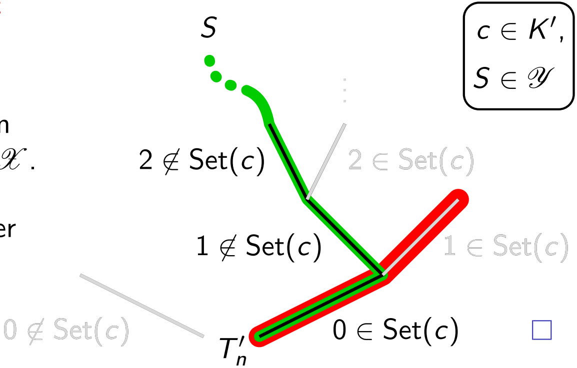
Making $\mathscr{Y} \cap \mathscr{Y}' = \mathscr{X}$ given $\mathscr{Y}$

$$\overbrace{\mathrm{I}\Delta_0^A \subseteq T_1 \subseteq T_2 \subseteq \cdots}^{(M,\mathscr{X})\text{-generic}} \; T_\omega \xrightarrow{\text{Henkin model}} K \xrightarrow{\text{coded sets}} \mathscr{Y}$$

$$\underbrace{\mathrm{I}\Delta_0^A \subseteq T'_1 \subseteq T'_2 \subseteq \cdots}_{(M,\mathscr{X},T_\omega)\text{-generic}} \; T'_\omega \longrightarrow K' \longrightarrow \mathscr{Y}'$$

Case 1: S is the only "consistent path". Then $S \in \Delta_1^0\text{-Def}(M, T'_n) \subseteq \mathscr{X}$.

Case 2: There is another "consistent path". Add this path to T'_n, so that $\mathrm{Set}(c) \neq S$.



□

Genericity and forcing

exists because $(M, \mathscr{X})$ is countable

Definition

A *condition* is $T \in \mathscr{X}$ satisfying $T \supseteq \mathrm{I}\Delta_0^A$ and $(M, \mathscr{X}) \models \mathrm{Con}(T)$.
A set $\mathscr{D}$ of conditions is *dense* if

$$\forall \text{condition } T \quad \exists \text{condition } T^* \supseteq T \quad T^* \in \mathscr{D}.$$

A sequence $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ of conditions is *$(M, \mathscr{X})$-generic* if

$$\forall (M, \mathscr{X})\text{-definable dense } \mathscr{D} \quad \exists n{<}\omega \quad T_n \in \mathscr{D}.$$

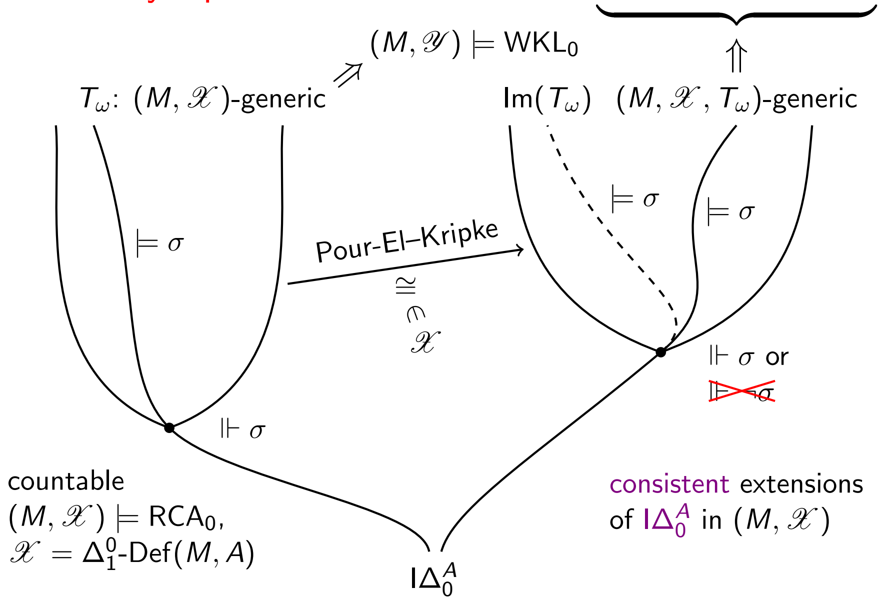
Truth Lemma

Assume $(M, \mathscr{X})$-genericity. Then for every $\mathscr{L}_{\mathrm{II}}(M, \mathscr{X})$-sentence σ,

$$(M, \mathscr{Y}) \models \sigma \quad \Leftrightarrow \quad \exists n{<}\omega \quad T_n \Vdash \sigma$$

every $(M, \mathscr{X})$-generic $T_0'' \subseteq T_1'' \subseteq T_2'' \subseteq \cdots \subseteq T_n \subseteq \cdots$
makes $(M, \mathscr{Y}'') \models \sigma$

Elementarily equivalent extensions with trivial intersection



Homogeneity

- direct formalization
- used $\mathrm{I}\Sigma^0_1$ and $\mathscr{X} = \Delta^0_1\text{-Def}(M, A)$

Theorem (Pour-El–Kripke 1967)

If T, T' are consistent extensions of $\mathrm{I}\Delta^A_0$ in $(M, \Delta^0_1\text{-Def}(M, A))$, then there is a bijection $F \in \Delta^0_1\text{-Def}(M, A)$ from formulas to formulas such that

$$(M, \mathscr{X}) \models \forall\varphi\ \big(\mathrm{Con}(T + \varphi) \leftrightarrow \mathrm{Con}(T' + F(\varphi))\big).$$

Proof

Back-and-forth plus fixed-point argument. $\square$

Further application

Arithmetized Completeness Theorem $\times$ forcing

Theorem (Simpson–Tanaka–Yamazaki 2002)

Every countable $(M, \mathscr{X}) \models \mathrm{RCA}_0$ can be extended to $(M, \Delta^0_1\text{-Def}(M, T_\omega)) \models \mathrm{RCA}_0$ for some $T_\omega \subseteq M$.

Proof sketch

$$\begin{tikzcd}
A \arrow[r, rightsquigarrow] & A_0, \arrow[d, bend right, "\text{coded in}"] & A_0 \oplus A_1, & A_0 \oplus A_1 \oplus A_2, & \dots \\
T_n \arrow[r, rightsquigarrow] & \langle T_0, S_0 \rangle, & \langle T_1, S_1 \rangle, & \langle T_2, S_2 \rangle, \arrow[u, bend right, "\Delta^0_1\text{-definable in}"'] & \dots
\end{tikzcd}$$

where S_n is an unbounded tree of consistent extensions of T_n. $\square$