An application of the Arithmetized Completeness Theorem to second-order arithmetic

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15 July, 2014

*Financial support from FWF Project P24654-N25 is acknowledged.

This talk

$\mathsf{WKL}_0 \supseteq \mathsf{I}\Sigma_1$

Harrington's Theorem

Every countable $M \models \mathsf{I}\Sigma_1$ can be expanded to $(M, \mathscr{X}) \models \mathsf{WKL}_0$.

Motivation

- Conservation results between theories
- Connections with end extensions
- Iterating Arithmetized-Completeness-Theorem constructions (joint research with Ali Enayat, Gothenburg)

Plan

1. Introduction
2. End extensions
3. Arithmetized Completeness Theorem
4. Further questions

First-order arithmetic

- $\mathscr{L}_{\mathrm{I}} = \{0, 1, +, \times, <\}$.
- $\Delta_0(\Sigma_1)$ is the smallest set of $\mathscr{L}_{\mathrm{I}}$-formulas that
  - contains all ~~atomic $\mathscr{L}_{\mathrm{I}}$ formulas~~ Σ_1-formulas; and
  - is closed under $\neg$, $\wedge$, $\vee$, and *bounded quantification*, i.e., $\forall v < t\ \cdots$ and $\exists v < t\ \cdots$.
- $\Sigma_1 = \{\exists \bar{v}\ \varphi(\bar{v}, \bar{x}) : \varphi \in \Delta_0\}$; its dual is called Π_1.
- Formulas equivalent to a Σ_1- and a Π_1-formula are called Δ_1.
- $\mathrm{I}\Delta_0$ is axiomatized by PA^- and the *induction scheme*

$$\theta(0) \wedge \forall x\ \big(\theta(x) \to \theta(x+1)\big) \to \forall x\ \theta(x),$$

  for $\theta \in \Delta_0$ possibly with parameters.
- $\mathrm{I}\Sigma_1$ is defined similarly.

Fact (Folklore)

$\mathrm{I}\Sigma_1$ proves the induction scheme for $\Delta_0(\Sigma_1)$-formulas.

Second-order arithmetic

- $\mathscr{L}_{\mathrm{II}} = \{0, 1, +, \times, <, \in\}$ has a *number sort* and a *set sort*.
- Δ^0_0 is the smallest set of $\mathscr{L}_{\mathrm{II}}$-formulas that
  - contains all atomic $\mathscr{L}_{\mathrm{II}}$-formulas; and
  - is closed under $\neg$, $\wedge$, $\vee$, and *bounded quantification*, i.e., $\forall v < t\ \cdots$ and $\exists v < t\ \cdots$.
- $\Sigma^0_1 = \{\exists \bar{v}\ \varphi(\bar{v}, \bar{x}, \bar{S}) : \varphi \in \Delta^0_0\}$; its dual is called Π^0_1.
- Formulas equivalent to a Σ^0_1- and a Π^0_1-formula are called Δ^0_1.
- WKL_0 is axiomatized by PA^-, extensionality,
  - induction scheme for Σ^0_1-formulas with set parameters;
  - Δ^0_1-*comprehension*, i.e., every Δ^0_1-definable class of numbers is a set; and
  - *Weak König Lemma* (WKL), i.e., every unbounded binary tree has an unbounded path.

Fact (Folklore)

If $M \models \mathrm{I}\Sigma_1$, then $(M, \Delta_1\text{-Def}(M)) \models \mathsf{WKL}_0 - \mathsf{WKL}$.

End extensions

Let $M \models \mathrm{I}\Sigma_1$ and $K \models \mathrm{I}\Delta_0$ properly *end extending* M, i.e.,

$$\forall x \in K \setminus M \quad \forall y \in M \quad x > y.$$

Say $c \in K$ *codes* $S \subseteq M$ if

$$S = \{i \in M : i\text{-th prime divides } c\}.$$

Define $\mathrm{Cod}(K/M) = \{S \subseteq M : S \text{ is coded by some } c \in K\}$.

Proposition (Folklore)

$(M, \mathrm{Cod}(K/M))$ satisfies WKL and Δ^0_1-comprehension.

Proof

Overspill. □

Making Σ^0_1-induction hold in $(M, \mathrm{Cod}(K/M))$

- Fix $M \models \mathrm{I}\Sigma_1$ and $K \models \mathrm{I}\Delta_0$ properly end extending M.
- Let $S \subseteq M$ coded by $c \in K$.

$(M, \mathrm{Cod}(K/M))$ $\qquad \exists v \quad \varphi(v, x, S) \qquad \varphi \in \Delta^0_0$

↓ replacing "$i \in S$" by "i-th prime divides c"

(K, M) $\qquad \exists v \in M \quad \varphi'(v, x, c) \qquad \varphi' \in \Delta_0$

↓ if M has enough control over K

M $\qquad \eta(x, \ldots) \qquad \eta \in \Delta_0(\Sigma_1)$

The Arithmetized Completeness Theorem (ACT)

Refinement of ACT for $\mathrm{I}\Sigma_1$ (Cornaros–Dimitracopoulos 2001)

In $M \models \mathrm{I}\Sigma_1$, every Δ_1-definable consistent $\mathscr{L}_{\mathrm{I}}$-theory $T \supseteq \mathrm{I}\Delta_0$ has a complete consistent (Henkinized) extension T^* in which

$$\{x \in M : T^* \vdash \varphi(\underline{a}, \underline{x}) \text{ for some } a \in M\}$$

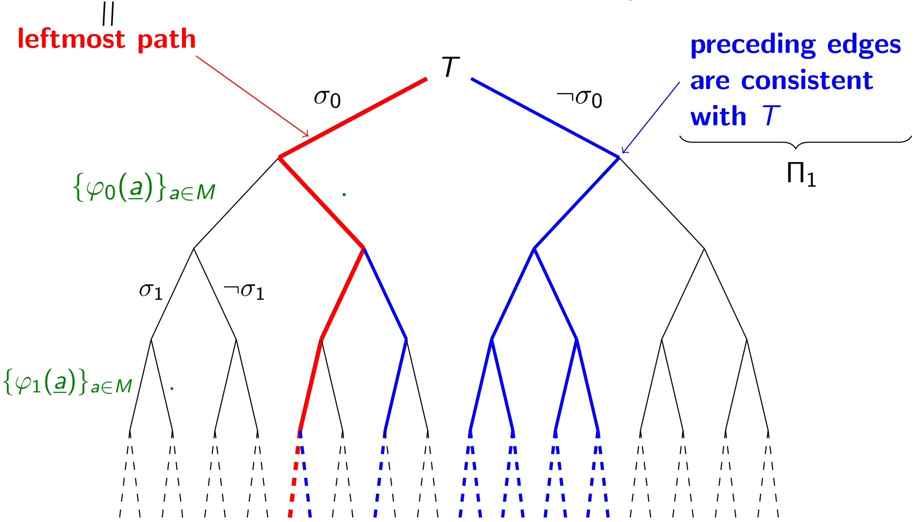
is $\Delta_0(\Sigma_1)$-definable in M for all formulas $\varphi(v, x)$.

ACT gives end extensions

- T^*, being Henkinized, has a term model K.
- K realizes $\underline{a} = 0 + \underbrace{1 + 1 + \cdots + 1}_{a\text{-many 1s}}$ for each $a \in M$; so $K \supseteq M$.
- K is an end extension of M, because for each $a \in M$,

$$\mathrm{I}\Delta_0 \vdash \forall x {<} \underline{a+1} \; (x = \underline{0} \vee x = \underline{1} \vee \cdots \vee x = \underline{a}).$$

A completion T^* of a Δ_1-definable theory T such that...



$$\{x \in M : T^* \vdash \varphi(\underline{a}, \underline{x}) \text{ for some } a \in M\} \in \Delta_0(\Sigma_1)\text{-Def}(M).$$

Summary

Harrington's Theorem (Hájek 1993 version)

Every ~~countable~~ $M \models \mathrm{I}\Sigma_1$ can be expanded to $(M, \mathscr{X}) \models \mathrm{WKL}_0$.

Outline of our proof

1. Using the Arithmetized Completeness Theorem, end extend M properly to $K \models \mathrm{I}\Delta_0$ over which M has enough control.
2. Take the coded subsets $\mathrm{Cod}(K/M)$ of M in K.
3. $(M, \mathrm{Cod}(K/M)) \models \mathrm{WKL}$ because K is an end extension of M.
4. $(M, \mathrm{Cod}(K/M)) \models \mathrm{I}\Sigma_1^0$ because M controls enough of K.

Questions

(a) Can this method expand $(M, S_i)_{i \in \mathbb{N}} \models \mathrm{I}\Sigma_1^0$?

(b) Can this method prove the conservativity of other principles?