The model theory of generic cuts

Tin Lok Wong

Ghent University, Belgium

Joint work with Richard Kaye (Birmingham)

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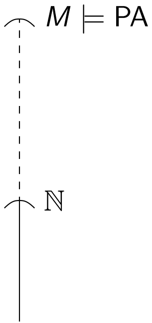
Existentially closed models



field

algebraically closed

Definition

A model M of a theory T is *existentially closed (ec)* if
for all $\exists_1$ formulas $\varphi(\bar{z})$ and all $\bar{c} \in M$,

if $K \models \varphi(\bar{c})$ for some $K \models T$ extending M,
then $M \models \varphi(\bar{c})$.

$\exists \bar{x}\ p(\bar{x}, \bar{c}) = 0$

Definition

Peano arithmetic (PA) consists of the axioms for *discretely ordered semirings* and the induction scheme for all first-order $\theta(x)$:

$$\theta(0) \wedge \forall x\ \big(\theta(x) \rightarrow \theta(x+1)\big) \rightarrow \forall x\ \theta(x).$$

Theorem (Rabin 1962)

There is no ec model of PA., *when* the language is $\{0, 1, +, \times, <\}$.

Skolem functions for $\mathscr{L}_\mathsf{A} = \{0, 1, +, \times, <\}$

Fix a recursive enumeration $(F_n)_{n\in\mathbb{N}}$ of all $\mathscr{L}_\mathsf{A}$ definable functions. We call these *Skolem functions*.

Proposition

With all these F_n's added to $\mathscr{L}_\mathsf{A}$, *every* model of PA is ec.

Every formula is quantifier-free.

The question

Can the theory of ec models be applied to models of PA *nontrivially*?

Yes!

Cuts

Let $M \models \mathrm{PA}$.

$F_n(x) = (\min y)(Y(x, y) \geqslant n)$,
where Y is an *indicator*

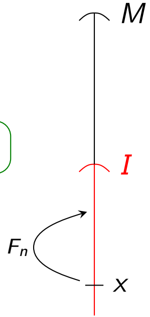
Definition

A *cut* of M is a nonempty proper initial segment of M that has no maximum element.

Definition

An *elementary cut* of M is a cut I such that $F_n(\bar{x}) \in I$ whenever $\bar{x} \in I$ and $n \in \mathbb{N}$.

elementary substructure



Why cuts?

- Model theory
- Independence results
- Second-order arithmetic
- Nonstandard analysis

Cuts are not definable

Definition

The language $\mathscr{L}_{\mathrm{cut}}$ contains

intended for a cut

- $0, 1, +, \times, <$,
- a new unary predicate symbol $\mathbb{I}$, and
- F_n for each $n \in \mathbb{N}$.

Definition

$\mathrm{PA}^{\mathrm{elem}}$ is the $\mathscr{L}_{\mathrm{cut}}$ theory consisting of

induction

- the axioms for discretely ordered semirings,
- the definitions for each Skolem function F_n,
- an axiom saying $\mathbb{I}$ is a cut, and
- $\forall \bar{x} \in \mathbb{I} \;\; F_n(\bar{x}) \in \mathbb{I}$ for every $n \in \mathbb{N}$.

Existentially closed elementary cuts

Question

Are there ec models of $\mathrm{PA}^{\mathrm{elem}}$?

Answer

Yes!

Theorem (∼Steinitz 1910)

Every model of an $\forall_2$ theory T extends to an ec model of T.

Proof

Extend repeatedly. □

Existential formulas in $\mathscr{L}_{\mathrm{cut}}$

Normal Form Lemma

Over $\mathrm{PA}^{\mathrm{elem}}$, every $\exists_1$ formula is equivalent to

$$\exists u \in \mathbb{I} \ \ \exists v \notin \mathbb{I} \ \ F_m(u, \bar{z}) = v$$

for some $m \in \mathbb{N}$.

Proof

Easy exercise. □

Corollary

Not every model of $\mathrm{PA}^{\mathrm{elem}}$ is ec.

$\mathrm{PA}^{\mathrm{elem}}$ is not model complete.

How do ec models of $\mathrm{PA}^{\mathrm{elem}}$ look like?

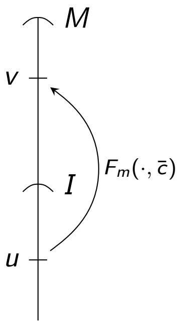
interpretation of $\mathbb{I}$

Note

A model $(M, I) \models \mathrm{PA}^{\mathrm{elem}}$ satisfies

$$\exists u \in \mathbb{I} \;\; \exists v \notin \mathbb{I} \;\; F_m(u, \bar{c}) = v$$

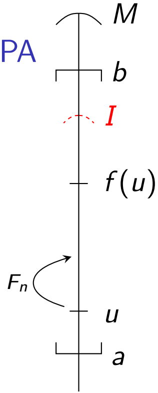
if and only if I is not closed under $u \mapsto F_m(u, \bar{c})$.



Slogan

A model $(M, I) \models \mathrm{PA}^{\mathrm{elem}}$ is existentially closed if and only if
I is closed under as *few* Skolem functions as possible.

A more concrete ec model (M, I) of $\mathrm{PA}^{\mathrm{elem}}$



Search I inside a fixed recursively saturated $M \models \mathrm{PA}$

Consider a Skolem function $f\colon M \to M$.

- Suppose I is to live between $a, b \in M$.
- We need $a \ll b$, i.e., $F_n(a) < b$ for all $n \in \mathbb{N}$.
- Either (a) $u \ll f(u)$ for some $u \in [a, b]$,
  or (b) $u \not\ll f(u)$ for all $u \in [a, b]$.
- If (a), then let I live between such u and $f(u)$.
- If (b), then any such I must be closed under f.

Start over in the smaller region with another f.

Theorem (Kaye–W)

Every countable recursively saturated $M \models \mathrm{PA}$ contains continuum many I such that (M, I) is an ec model of $\mathrm{PA}^{\mathrm{elem}}$.

Model companions

Theorem (Kaye–W)

$\mathrm{PA}^{\mathrm{elem}}$ has no *model companion*,

i.e., there is no $\mathscr{L}_{\mathrm{cut}}$ theory $T \supseteq \mathrm{PA}^{\mathrm{elem}}$ such that

(a) every model of $\mathrm{PA}^{\mathrm{elem}}$ extends to a model of T; and

(b) a model of $\mathrm{PA}^{\mathrm{elem}}$ is ec if and only if it satisfies T.

Proof

Ec models of $\mathrm{PA}^{\mathrm{elem}}$ are *never* $\forall_1$ recursively saturated. □

Example

The theory of algebraically closed fields is the model companion of the theory of fields.

Every nonconstant polynomial has a zero.

Conclusion

What we achieved

- Established that the elementary cuts have a nice theory of ec models and model companions.
- Obtained a good understanding of the $\exists_1$ formulas in $\mathscr{L}_{\mathsf{cut}}$.

Future work

- Understand the definable sets in a model of $\mathsf{PA}^{\mathsf{elem}}$.
- Classify the ec models of $\mathsf{PA}^{\mathsf{elem}}$.
- Investigate further model theoretic properties of $\mathsf{PA}^{\mathsf{elem}}$, e.g., existential closures, JEP and AP.