Generic cuts of models of Peano Arithmetic

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Overall plan

1. Arithmetic saturation
2. Pregeneric intervals
3. Generic cuts

Basic definitions

- *$\mathscr{L}_{\mathsf{A}}$* is the usual language for arithmetic $\{+, \times, <, 0, 1\}$.
- *Peano Arithmetic* (PA) is a set of basic axioms stating the usual algebraic properties of $+$, $\times$, $<$, 0 and 1, together with the induction scheme on all $\mathscr{L}_{\mathsf{A}}$-formulas.

Induction axiom on $\varphi(x, \bar{z})$

$$\forall \bar{z} \big[\varphi(0, \bar{z}) \wedge \forall x \big(\varphi(x, \bar{z}) \to \varphi(x+1, \bar{z}) \big) \to \forall x \varphi(x, \bar{z}) \big].$$

Arithmetic saturation

Definition

Let A and B be subsets of $\mathbb{N}$.
We say that A is *arithmetic* in B if

there is an arithmetical formula $\Psi(x, X)$ such that

$$A = \{n \in \mathbb{N} : \mathbb{N} \models \Psi(n, B)\}.$$

Definition

A model M of PA is *arithmetically saturated* if
every type arithmetic in $\mathrm{tp}(\bar{a})$ for some $\bar{a} \in M$ is realized.

Fact

Countable arithmetically saturated models of PA are homogeneous.

A vague notion of distance

Fix a countable arithmetically saturated model M of PA.

Definition

An *elementary cut* of M is an elementary substructure I that is closed downwards, i.e.,

if $x \in M$ and $y \in I$ such that $x < y$, then $x \in I$.

Definition

A closed interval $[a, b]$ in M is an *elementary interval* if there is an elementary cut I such that $a \in I < b$.

Pregeneric intervals

Definition

Let $\bar{c} \in M$ and $[a, b]$ be an elementary interval.
We say that $[a, b]$ is *pregeneric over $\bar{c}$* if

> for every elementary subinterval $[u, v]$ of $[a, b]$
> there is an elementary subinterval $[a', b']$ of $[u, v]$
> such that $(M, a, b, \bar{c}) \cong (M, a', b', \bar{c})$.

Theorem

Let M be a countable arithmetically saturated model of PA, and $\bar{c} \in M$.
Then every elementary interval $[a, b]$ contains a pregeneric elementary subinterval over $\bar{c}$.

Sketch of proof.

By a tree argument. □

Generic cuts

Definition

An elementary cut I is *generic* if
it is contained in a pregeneric interval over $\bar{c}$ for every $\bar{c} \in M$.

Theorem

Let M be a countable arithmetically saturated model of PA,
I be a generic cut, $\bar{c} \in M$, and
$[a, b]$ be an elementary interval containing I
such that $[a, b]$ is pregeneric over $\bar{c}$.
Then for all generic cuts J contained in $[a, b]$,

$$(M, I, \bar{c}) \cong (M, J, \bar{c}).$$

Sketch of proof.

Use a back-and-forth argument. □

Some remarks

- A converse to this theorem also holds.
- The set of generic cuts is *canonical* in the sense that it is *the* smallest comeagre set in the space of elementary cuts that is closed under automorphisms of M.

An application

Corollary

Let M be a countable arithmetically saturated model of PA,
and I be a generic cut of M.
Then for all tuples $\bar{c}, \bar{d} \in M$,

$$(M, I, \bar{c}) \cong (M, I, \bar{d})$$

if and only if

- $\mathrm{tp}^M(\bar{c}) = \mathrm{tp}^M(\bar{d})$, and
- for every $\mathscr{L}_{\mathsf{A}}$-formula $\varphi(x, \bar{z})$,

$$(\max x \in I)(M \models \varphi(x, \bar{c})) \text{ exists}$$

$$\Updownarrow$$

$$(\max x \in I)(M \models \varphi(x, \bar{d})) \text{ exists.}$$

In particular, the expanded structure (M, I) is homogeneous.

What can generic cuts give us?

- A new example of *free cuts*.
- ‘Quantifier elimination’ perhaps?
- A rich automorphism group $\mathrm{Aut}(M, I)$.