Generic cuts in models of Peano arithmetic

Tin Lok Wong

University of Birmingham, United Kingdom

Joint work with Richard Kaye (Birmingham)

25 June, 2009

Preliminary definitions

- $\mathscr{L}_{\mathrm{I}}$ is the first-order language for arithmetic $\{0, 1, +, \times, <\}$.
- *Peano Arithmetic* (PA) is the $\mathscr{L}_{\mathrm{I}}$-theory consisting of
  axioms for the non-negative part of discretely ordered rings
  and the *induction axiom*

  $$\forall \bar{z}\big[\varphi(0, \bar{z}) \wedge \forall x\big(\varphi(x, \bar{z}) \rightarrow \varphi(x+1, \bar{z})\big) \rightarrow \forall x \varphi(x, \bar{z})\big].$$

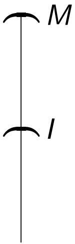
  for each $\mathscr{L}_{\mathrm{I}}$-formula $\varphi(x, \bar{z})$.

Aim

Understand structures of the form

$$(M, I)$$

where $M \models \mathrm{PA}$ and I is cut of M.



- How complicated is $\mathrm{Th}(M, I)$ in relation to $\mathrm{Th}(M)$?
- How does $\mathrm{Aut}(M, I)$ sit inside $\mathrm{Aut}(M)$?
- Is (M, I) easier to study than $(I, \mathrm{SSy}_I(M))$ where

$$\mathrm{SSy}_I(M) = \{X \cap I : X \subseteq M \text{ is definable with parameters}\}?$$

Arithmetic saturation

Definition

A model M of PA is *recursively saturated* if
every recursive type over M is realized in M.

Fact

Countable recursively saturated models of PA are ω-homogeneous.

Definition

A model M of PA is *arithmetically saturated* if
it is recursively saturated and $(\mathbb{N}, \mathsf{SSy}_{\mathbb{N}}(M)) \models \mathsf{ACA}_0$.

Topological background

Fix a countable arithmetically saturated model M of PA.

Definition

A cut of M is *elementary* if it is an elementary substructure of M.
We write $I \prec_{\mathrm{e}} M$ for 'I is an elementary cut of M.'

Definition

An *elementary interval* is a *nonempty* set of the form

$$[\![a, b]\!] = \{I \prec_{\mathrm{e}} M : a \in I < b\}$$

where $a, b \in M$.

Facts

- The elementary intervals generate a topology on the collection of all elementary cuts.
- The space of elementary cuts is homeomorphic to the Cantor set.

Genericity

Definition

A subset of a topological space is *comeagre* if
it contains a countable intersection of dense open sets.

Definition

An elementary cut is *generic* if
it is contained in any comeagre set of elementary cuts
that is closed under the automorphisms of M.

Pregeneric intervals

Theorem

Let $c \in M$ and $[\![a, b]\!]$ be an elementary interval. Then there is an elementary subinterval $[\![r, s]\!]$ of $[\![a, b]\!]$ such that

> for every elementary subinterval $[\![u, v]\!]$ of $[\![r, s]\!]$
> there is an elementary subinterval $[\![r', s']\!]$ of $[\![u, v]\!]$
> such that $(M, r, s, c) \cong (M, r', s', c)$.

This subinterval $[\![r, s]\!]$ is said to be *pregeneric over c*.

Proof.

A tree argument. □

Generic cuts

Take an enumeration $(c_n)_{n\in\mathbb{N}}$ of M.
Starting with an arbitrary elementary interval $[\![a_0, b_0]\!]$,
construct a sequence $[\![a_0, b_0]\!] \supseteq [\![a_1, b_1]\!] \supseteq [\![a_2, b_2]\!] \supseteq \cdots$
such that $[\![a_{n+1}, b_{n+1}]\!]$ is pregeneric over c_n for all $n \in \mathbb{N}$.
Then there is a unique elementary cut in $\bigcap_{n\in\mathbb{N}}[\![a_n, b_n]\!]$.

Theorem

The cuts constructed in this way are exactly the generic cuts.

Proof.

Back-and-forth. □

Generic cuts under automorphisms

Proposition

$(M, I_1) \cong (M, I_2)$ for all generic cuts I_1, I_2 in M.

Theorem

If I is a generic cut of M and $c, d \in I$ such that

$$\mathrm{tp}(c) = \mathrm{tp}(d),$$

then

$$(M, I, c) \cong (M, I, d).$$

Description of truth

Theorem

Let I be a generic cut of M.
Then for all $c, d \in M$,

$$(M, I, c) \cong (M, I, d)$$

if and only if

- $\text{tp}(c) = \text{tp}(d)$, and
- for every $\mathscr{L}_{\text{I}}$-formula $\varphi(x, z)$,

$$\{x \in I : M \models \varphi(x, c)\} \text{ has an upper bound in } I$$

$$\Updownarrow$$

$$\{x \in I : M \models \varphi(x, d)\} \text{ has an upper bound in } I.$$

Conclusion

What we did

- Picked out a more tractable (M, I)
  for each countable arithmetically saturated model M.
- Obtained some information about
  the automorphisms of this (M, I).
- Understood more about the fine structure of
  countable arithmetically saturated models.

What next?

Let I be a generic cut.

- What is special about $(I, \mathrm{SSy}_I(M))$ and $\mathrm{Th}(M, I)$?
- How does $\mathrm{Aut}(M, I)$ sit inside $\mathrm{Aut}(M)$?
- How does (M, I) interact with other (M, J) where $J \prec_{\mathrm{e}} M$?