Generic cuts in models of Peano arithmetic

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Joint work with Richard Kaye (Birmingham)

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Preliminary definitions

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- $\mathscr{L}_\mathsf{A}$ is the first-order language for arithmetic $\{0, 1, +, \times, <\}$.
- *Peano Arithmetic* (PA) is the $\mathscr{L}_\mathsf{A}$-theory consisting of
  axioms for the non-negative part of discretely ordered rings
  and the *induction axiom*

  $$\forall \bar{z} \big[\varphi(0, \bar{z}) \land \forall x \big(\varphi(x, \bar{z}) \to \varphi(x + 1, \bar{z}) \big) \to \forall x \varphi(x, \bar{z}) \big].$$

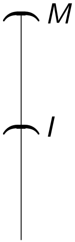
  for each $\mathscr{L}_\mathsf{A}$-formula $\varphi(x, \bar{z})$.

Aim

Understand structures of the form

$$(M, I)$$

where $M \models \mathrm{PA}$ and I is cut of M.



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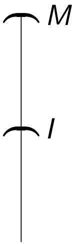

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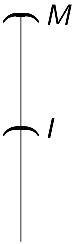

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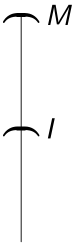

- How complicated is $\mathrm{Th}(M, I)$ in relation to $\mathrm{Th}(M)$?
- How does $\mathrm{Aut}(M, I)$ sit inside $\mathrm{Aut}(M)$?
- Is (M, I) easier to study than $(I, \mathrm{SSy}_I(M))$ where

$$\mathrm{SSy}_I(M) = \{X \cap I : X \subseteq M \text{ is definable with parameters}\}?$$

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A model M of PA is *arithmetically saturated* if
it is recursively saturated and $(\mathbb{N}, \mathsf{SSy}_{\mathbb{N}}(M)) \models \mathsf{ACA}_0$.

Topological background

Fix a countable arithmetically saturated model M of PA.

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We write $I \prec_{\mathrm{e}} M$ for 'I is an elementary cut of M.'

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Fact

The elementary intervals generate a topology on the collection of all elementary cuts.

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Fact

The space of elementary cuts is homeomorphic to the Cantor set.

Genericity

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A generic cut satisfies
all ‘generic’ properties.

Pregeneric intervals

Theorem

Let $c \in M$ and $[\![a, b]\!]$ be an elementary interval. Then there is an elementary subinterval $[\![r, s]\!]$ of $[\![a, b]\!]$ such that

for every elementary subinterval $[\![u, v]\!]$ of $[\![r, s]\!]$
there is an elementary subinterval $[\![r', s']\!]$ of $[\![u, v]\!]$
such that $(M, r, s, c) \cong (M, r', s', c)$.

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Proof.

A tree argument. □

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Back-and-forth. □

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$(M, I_1) \cong (M, I_2)$ for all generic cuts I_1, I_2 in M.

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If I is a generic cut of M and $c, d \in I$ such that

$$\mathrm{tp}(c) = \mathrm{tp}(d),$$

then

$$(M, I, c) \cong (M, I, d).$$

Description of truth

Theorem

Let I be a generic cut of M.
Then for all $c, d \in M$,

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if and only if

- $\mathrm{tp}(c) = \mathrm{tp}(d)$, and
- for every $\mathscr{L}_{\mathsf{A}}$-formula $\varphi(x, z)$,

$$\{x \in I : M \models \varphi(x, c)\} \text{ has an upper bound in } I$$

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Quantifier elimination?

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- Obtained some information about the automorphisms of this (M, I).
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Let I be a generic cut.

- What is special about $(I, \mathrm{SSy}_I(M))$ and $\mathrm{Th}(M, I)$?
- How does $\mathrm{Aut}(M, I)$ sit inside $\mathrm{Aut}(M)$?
- Investigate the existential closure properties of (M, I).