End-extending models of arithmetic using Ramsey-type principles

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Plan

- Introduction
- Two notions of strength for end-extensions
- A construction using the Regularity Scheme
- A construction using Arithmetical Ramsey's Theorem
- Future work

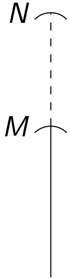
Preliminary definitions

- $\mathscr{L}_{\mathrm{I}}$ is the first-order language of arithmetic $\{0, 1, +, \times, <\}$.
- *Peano Arithmetic* (PA) is the $\mathscr{L}_{\mathrm{I}}$ theory consisting of
  axioms for the non-negative part of discretely ordered rings
  and the *induction axiom*

$$\varphi(0) \land \forall x \big(\varphi(x) \to \varphi(x+1)\big) \to \forall x \varphi(x)$$

  for each $\mathscr{L}_{\mathrm{I}}$ formula $\varphi(x)$.
- Informally, 'model of arithmetic' refers to
  an $\mathscr{L}_{\mathrm{I}}$ structure satisfying just enough induction axioms.

End-extensions



Definition

An *end-extension* of an $\mathscr{L}_{\mathrm{I}}$ structure M is an extension N where all new elements are added above the old ones, i.e.,

$$\forall m \in M \; \forall n \in N \; \big(n < m \Rightarrow n \in M\big).$$

Aim

To understand the role of Ramsey principles in end-extension constructions

- better understanding of Ramsey principles
- model theoretic view of reverse mathematics
- applications in nonstandard analysis

Elementarity

Let N be an extension of a model M of arithmetic.

Definition

Let Φ be a class of formulas.
The extension N of M is *Φ-elementary* if
for all $\varphi(\bar{x}) \in \Phi$ and all $\bar{a} \in M$,

$$N \models \varphi(\bar{a}) \Leftrightarrow M \models \varphi(\bar{a}).$$

The 'amount of elementarity' of the extension N of M
is the set of $\varphi(\bar{x})$ such that

$$\forall \bar{a} \in M \left(N \models \varphi(\bar{a}) \Leftrightarrow M \models \varphi(\bar{a})\right).$$

Language of second-order arithmetic

- $\mathscr{L}_{\mathrm{II}}$ denotes the language of second-order arithmetic.
- Second-order arithmetic has a number sort and a set sort.
- Lower case letters $x, y, z, \ldots$ are used for number variables.
- Upper case letters $X, Y, Z, \ldots$ are used for set variables.
- There is a binary relation symbol $\in$ intended for membership between a number and a set.

Example

The axiom of extensionality

$$\forall X \forall Y \big(\forall v (v \in X \leftrightarrow v \in Y) \rightarrow X = Y\big)$$

can be expressed in the language of second-order arithmetic.

Models of second-order arithmetic

- A model of second-order arithmetic $(M, \mathscr{X})$ consists of
  - a number universe M, over which the number quantifiers quantify; and
  - a set universe $\mathscr{X}$, over which the set quantifiers quantify.
- Without loss, assume $\mathscr{X} \subseteq \mathscr{P}(M)$.
- Sets in $\mathscr{X}$ are called *internal sets*.

Example

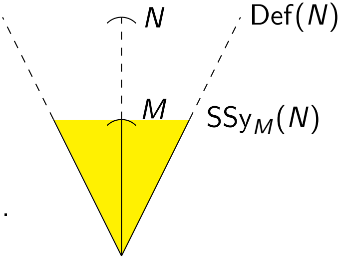
Let $\mathsf{Def}(\mathbb{N})$ denote the class of $\mathscr{L}_{\mathrm{I}}$ definable subsets of $\mathbb{N}$. Then $(\mathbb{N}, \mathsf{Def}(\mathbb{N}))$ is a model of second-order arithmetic.

Second-order content

Let N be an end-extension of a model M of arithmetic.

Definition (Tennenbaum 1959, Friedman 1973, Kirby–Paris 1976, Paris 1979, Hirst 1987, ...)

$$\mathrm{SSy}_M(N) = \{X \cap M : X \in \mathrm{Def}(N)\}.$$



The 'second-order content' of the end-extension N of M is

$$\mathrm{Th}\big((M, \mathrm{SSy}_M(N))\big).$$

Theories of second-order arithmetic

The Big Five theories in reverse mathematics are

- RCA_0,
- WKL_0,
- ACA_0,
- ATR_0, and
- $\Pi^1_1\text{-}\mathsf{CA}_0$,

in increasing order of strength.

Between the two notions

Theorem (Phillips 1974, Gaifman 1976, Paris–Kirby 1977)

Let M be a model of arithmetic.
Then M has a proper $\mathscr{L}_{\mathrm{I}}$ elementary end-extension
if and only if it has a proper end-extension N such that

$$(M, \mathsf{SSy}_M(N)) \models \mathsf{ACA}_0.$$

Remark

For $n \geqslant 2$, a similar correspondence holds
between Σ_n-elementarity and the collection scheme $\mathrm{B}\Sigma_n^*$.

Ramsey's Theorem

- Write $X \subseteq_{\mathsf{cf}} Y$ to mean 'X is an unbounded subset of Y'.
- Denote the class of r-element subsets of a set S by $[S]^r$.

Ramsey's Theorem (1930)

Let $r, k \in \mathbb{N}$ and $S \subseteq_{\mathsf{cf}} \mathbb{N}$.
For all colourings $c\colon [S]^r \to k$,
there exists $H \subseteq_{\mathsf{cf}} S$ such that $c \restriction [H]^r$ is constant.

Formalized Ramsey's Theorem

Let S be an unbounded definable subset.
For all $\mathscr{L}_{\mathrm{I}}$ formulas $\theta(x_1, x_2, \ldots, x_r, m)$ and all k,
there exists a definable $H \subseteq_{\mathsf{cf}} S$ such that

$$\forall \bar{x} \in [S]^r \, \exists m < k \, \theta(\bar{x}, m) \to \exists m < k \, \forall \bar{x} \in [H]^r \, \theta(\bar{x}, m).$$

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Regularity Scheme

Let S be an unbounded definable subset.
For all $\mathscr{L}_{\mathrm{I}}$ formulas $\theta(x, m)$ and all k,
there exists a definable $H \subseteq_{\text{cf}} S$ such that

$$\forall x \in S\, \exists m < k\ \theta(x, m) \to \exists m < k\ \forall x \in H\ \theta(x, m).$$

Arithmetical Ramsey's Theorem

Definition

An *arithmetical* formula is an $\mathscr{L}_{\mathrm{II}}$ formula without set quantifiers.
The set of arithmetical formulas is denoted by $\mathscr{L}_{\mathrm{I}}^*$.

Arithmetical Ramsey's Theorem (Galvin–Prikry 1973)

Let $k \in \mathbb{N}$ and $S \subseteq_{\mathsf{cf}} \mathbb{N}$.
For all *arithmetically definable* colourings $c\colon [S]^{\mathbb{N}} \to k$,
there exists $H \subseteq_{\mathsf{cf}} S$ such that $c \restriction [H]^{\mathbb{N}}$ is constant.

Formalized Arithmetical Ramsey's Theorem (ART)

Let S be an unbounded internal set.
For all arithmetical formulas $\theta(X, m)$ and all k,
there exists an internal $H \subseteq_{\mathsf{cf}} S$ such that

$$\forall X \subseteq_{\mathsf{cf}} S\, \exists m < k\, \theta(X, m) \to \exists m < k\, \forall X \subseteq_{\mathsf{cf}} H\, \theta(X, m).$$

Strength of Ramsey principles

Theorem

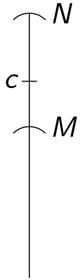
Every model of PA satisfies Formalized Ramsey's Theorem.

Theorem (Simpson 1981)

$\mathsf{RCA}_0 + \mathsf{ART}$ proves Π^1_1-CA_0.

Adjoining a new number

Keisler 1970, Phillips 1974, Gaifman 1976, . . .



Let $M \models \mathrm{PA}$ be *countable*.

Aim

To build a proper *elementary* end-extension N of M.

Plan

- Adjoin an 'infinite' element c to M
  so that N is generated by $M \cup \{c\}$.
- Control the $\mathscr{L}_{\mathrm{I}}(M)$ type $p(c)$ of c in N
  so that N is an end-extension.

Conditions on $p(c)$

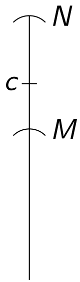
- $p(c) \supseteq \mathrm{Th}(M)$.
- $p(c) \supseteq \{c > a : a \in M\}$.
- If $\varphi(c) \in p(c)$, then $M \models \forall y \exists x > y\ \varphi(x)$.
- Avoid the following:

  There is an $\mathscr{L}_{\mathrm{I}}(M)$ formula $\theta(x, m)$ and $k \in M$ such that $p(c)$ contains

  (a) $\exists m < k\, \theta(c, m)$, and

  (b) $\forall m(\theta(c, m) \rightarrow m \neq m')$ for all $m' < k$ in M.

  $\Big\}\quad (*)$



Applying Regularity

- Recursively construct a sequence of unbounded definable sets

$$S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$$

- The set of defining formulas for $S_0, S_1, S_2, \ldots$ constitute $p(x)$.
- At infinitely many stages, we make $(*)$ fail for a pair

$$\langle \theta(x, m), k \rangle.$$

- If $M \models \forall x \in S_n \, \exists m < k \; \theta(x, m)$,
  then pick a definable $S_{n+1} \subseteq_{\text{cf}} S_n$ using Regularity such that

$$M \models \exists m < k \; \forall x \in S_{n+1} \; \theta(x, m).$$

- If $M \not\models \forall x \in S_n \, \exists m < k \; \theta(x, m)$, then $S_{n+1} = S_n$.

The outcome

Theorem

N is an elementary end-extension of M.

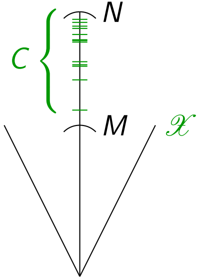
Theorem

Using Formalized Ramsey's Theorem in place of Regularity, one gets an end-extension N such that

$$(M, \mathrm{SSy}_M(N)) \models \mathrm{ACA}_0.$$

Adjoining a new set

Kaye–W.



Let $(M, \mathscr{X}) \models \mathrm{RCA}_0 + \mathrm{ART}$ be *countable*.

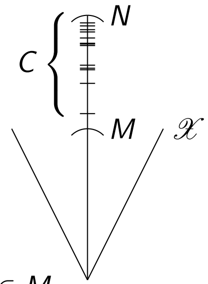
Aim

Build a stronger end-extension N of M using ART.

Plan

- Adjoin a new set C to $(M, \mathscr{X})$
  so that N is generated by M with the predicates $\mathscr{X} \cup \{C\}$.
- Control the $\mathscr{L}_{\mathrm{I}}^*(M, \mathscr{X})$ type $p(C)$ of C in N
  so that N is an end-extension.

Conditions on $p(C)$



- $p(C) \supseteq \mathrm{Th}_{\mathscr{L}_{\mathrm{I}}^*}(M, \mathscr{X})$.
- We have no idea where C should be.
- Avoid the following:

There is an $\mathscr{L}_{\mathrm{I}}^*(M, \mathscr{X})$ formula $\theta(X, m)$ and $k \in M$ such that $p(C)$ contains

(a) $\exists m < k\, \theta(C, m)$, and

(b) $\forall m(\theta(C, m) \rightarrow m \neq m')$ for all $m' < k$ in M.

$$\left.\right\} \quad (\dagger)$$

Applying ART

- Recursively construct a sequence of unbounded internal sets

$$S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$$

- C is an unbounded 'subset' of each of $S_0, S_1, S_2, \ldots$
- At infinitely many stages, we make (†) fail for a pair

$$\langle \theta(X, m), k \rangle.$$

- If $(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_n \, \exists m < k \, \theta(X, m)$,
  then pick an internal $S_{n+1} \subseteq_{\mathrm{cf}} S_n$ using ART such that

$$(M, \mathscr{X}) \models \exists m < k \, \forall X \subseteq_{\mathrm{cf}} S_{n+1} \, \theta(X, m).$$

- If $(M, \mathscr{X}) \not\models \forall X \subseteq_{\mathrm{cf}} S_n \, \exists m < k \, \theta(X, m)$, then $S_{n+1} = S_n$.

The outcome

Theorem

N is an elementary end-extension of M.

Theorem

If, in addition, $(M, \mathscr{X})$ satisfies UART,
then one can make $(M, \mathrm{SSy}_M(N)) \models \Pi^1_1\text{-CA}_0$.

Exact strength of ART

Uniform Arithmetical Ramsey's Theorem (UART)

Let S be an unbounded internal set.
For all arithmetical formulas $\theta_i(X, m)$ and all k,
there exists an internal sequence $H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots$
of unbounded subsets of S such that

$$\forall i \, \forall X \subseteq_{\mathrm{cf}} S \, \exists m < k \, \theta_i(X, m) \rightarrow \forall i \, \exists m < k \, \forall X \subseteq_{\mathrm{cf}} H_i \, \theta_i(X, m).$$

Question

Does $\mathrm{RCA}_0 + \mathrm{ART}$ prove UART?

Question

Does $\Pi^1_1\text{-}\mathrm{CA}_0$ prove ART?

Future work

Question

Is the use of ART and UART necessary?

Question

What amount of elementarity corresponds to end-extensions stronger than ACA_0?

Problem

Find a construction that characterizes ATR_0.

Problem

Generalize the arguments to other combinatorial principles.