Cuts closed under a specified family of functions

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23 September, 2012

*My current appointment is funded by the John Templeton Foundation.

Main theme

In a model of arithmetic,
control under which functions
an initial segment is closed.

Plan of talk

- Nonstandard models of arithmetic
- Closing under a function
- Avoiding closing under a function
- Reflections

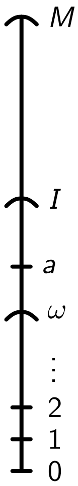
Nonstandard arithmetic

- The *language for arithmetic* is $\{0, 1, +, \times, <\}$.
- *Peano arithmetic* (PA) consists of the axioms for *discretely ordered semirings*, and the *induction axiom*

$$\theta(0) \wedge \forall x \big(\theta(x) \rightarrow \theta(x+1)\big) \rightarrow \forall x \ \theta(x)$$

  for each formula $\theta(x)$.
- A *nonstandard* model of PA is a model not isomorphic to ω.
- Skolem (1934) showed that nonstandard models exist.

Cuts



Fix a nonstandard model $M \models \mathrm{PA}$.

Definition

A *cut* of M is a nonempty proper initial segment of M that is closed under $x \mapsto x + 1$.

Example

ω is the *standard cut* of M.

Definition

An element $a \in M$ is *nonstandard* if $a > \omega$.

Closing under definable functions

Definition

A function $F\colon M \to M$ is *definable* if
there is a formula $\chi(x, y, z)$ and $c \in M$ such that

$$F(x) = y \quad \Leftrightarrow \quad M \models \chi(x, y, c)$$

for all $x, y \in M$.

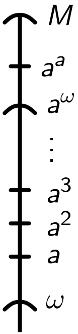
Problem

How do we make cuts that are closed under a definable function F?

Assumption

All our functions F satisfy $x \leqslant F(x) \leqslant F(x+1)$ for all $x \in M$.

Closing under multiplication



Constructing a cut that is closed under $\times$

- Pick any nonstandard $a \in M$.
- Let $a^\omega = \sup\{a^k : k \in \omega\}$.
- Then a^ω is closed under $\times$.
- Notice a^ω is *not* closed under $x \mapsto x^x$.

Primitive recursive functions

Definition (Grzegorczyk)

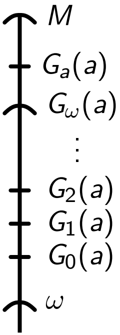
Set
$$G_0(x) = x + 1,$$
$$G_{k+1}(x) = G_k^{(x)}(x) \qquad \text{for all } k, x.$$

Fact

There is a formula $\chi(k, x, y)$ representing $G_k(x) = y$.

Constructing a cut that is closed under the G_k's

- Pick any nonstandard $a \in M$.
- Let $G_\omega(a) = \sup\{G_k(a) : k \in \omega\}$.
- Then $G_\omega(a)$ is closed under G_k for all $k \in \omega$.
- Notice $G_\omega(a)$ is not closed under $x \mapsto G_x(x)$.



Domination

Question (informal)

Is $G_\omega(a)$ closed under any function "other than" the G_k's?

Definition

Let $F, G \colon M \to M$ and I be a cut. Then *F dominates G on I* if $F(x) \geqslant G(x)$ for all large enough $x \in I$.

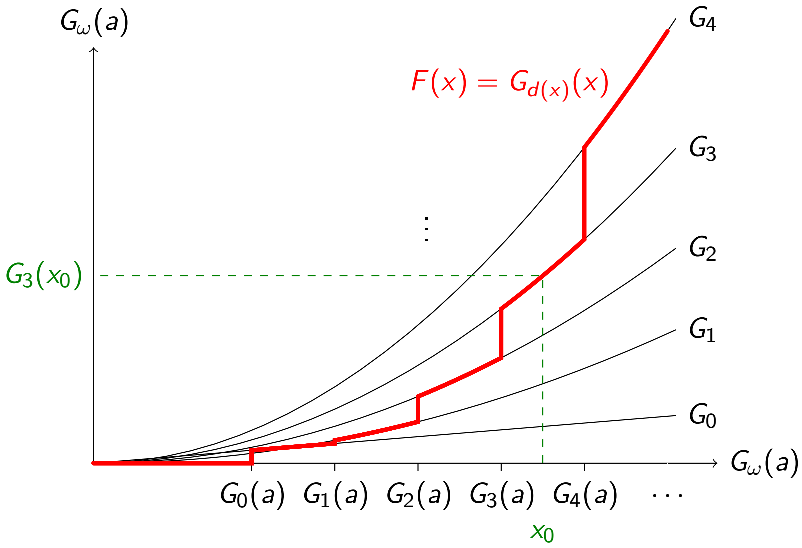
Question (formal)

Is there a definable $F \colon M \to M$ under which $G_\omega(a)$ is closed that dominates G_k on $G_\omega(a)$ for every $k \in \omega$?

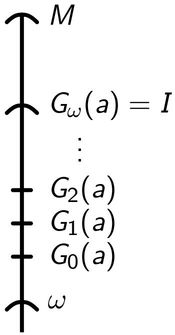
Answer

Yes!

Diagonalization



What we know about $I = G_\omega(a)$



Summary

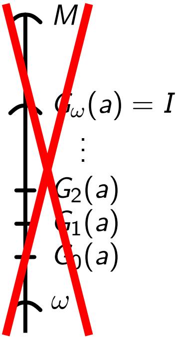
- I is the smallest cut that contains a and is closed under G_k for all $k \in \omega$.
- There exists a definable function $F\colon M \to M$ under which I is closed but dominates G_k on I for every $k \in \omega$.

Fact

The following are equivalent for a definable function $F\colon M \to M$.

- I is closed under F.
- F is dominated on I by $x \mapsto G_{d(x)}(x)$ for some definable function $d\colon M \to M$ that satisfies $d(I) = \omega$.

Closing exclusively



Definition

A cut I is closed *exclusively* under the G_k's if

- I is closed under G_k for every $k \in \omega$, and
- every definable function under which I is closed is dominated by G_k on I for some $k \in \omega$.

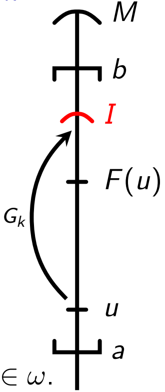
Question

Are there cuts that are closed exclusively under the G_k's?

Answer

Yes, at least when M is countable.

A cut I that is closed exclusively under the G_k's



Search I in a countable nonstandard $M \models \mathrm{PA}$

Consider a definable $F\colon M \to M$.

- Suppose I is to live between $a, b \in M$.
- We need $a \ll b$, i.e., $G_k(a) < b$ for all $k \in \omega$.

(a) Suppose $u \ll F(u)$ for some $u \in [a, b]$.

Then let I live between such u and $F(u)$.

(b) Suppose $u \not\ll F(u)$ for all $u \in [a, b]$.

Then $\max\left\{(\min k)(G_k(u) > F(u)) : u \in [a, b]\right\} \in \omega$.

So F is dominated by G_k on I for some $k \in \omega$.

Repeat with another definable function F' inside $[a', b']$.

Theorem

Every countable nonstandard model of PA contains continuum many cuts that are closed exclusively under the G_k's.

Existentially closed models

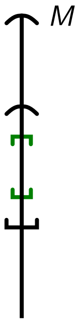
- Existentially closed models are a counterpart of algebraically closed fields in model theory.
- They are models that satisfy a maximal number of $\exists$ formulas.
- "A cut I not being closed under a function F" is existential:

$$\exists x \in I \ \ \exists y > I \ \ y = F(x).$$

Theorem
A cut I of M is closed exclusively under the G_k's if and only if (M, I) is an existentially closed model of the theory

$$\begin{aligned} &\mathrm{PA} + \{I \text{ is a cut}\} \\ &\quad + \{\forall x \in I \ \ \exists y \in I \ \ y = G_k(x) : k \in \omega\}. \end{aligned}$$

Constant regions



- Recall $a \ll b$ means $G_k(a) < b$ for all $k \in \omega$.
- We interpret $a \ll b$ as "$[a, b]$ is large".
- Let I be a cut that is closed exclusively under the G_k's.
- Then M is homogeneous at I, in the sense that every formula θ that is satisfied arbitrarily close to I is satisfied *densely* in a neighbourhood of I with respect to $\ll$.

Question

Can the model M be homogeneous in a larger region?

Answer (Kaye, W)

Yes, when M is countable arithmetically saturated.

Conclusion

What we saw

- There is a smallest cut $G_\omega(a)$ that contains a given $a \in M$ and is closed under a given definable family $(G_k)_{k\in\omega}$ of functions.
- This $G_\omega(a)$ is *not* closed *exclusively* under the G_k's.
- There exist cuts that are closed exclusively under the G_k's.
- This property is equivalent to being *existentially closed*.

Future work

- Automorphism group
- Independence results