MODEL THEORY OF ARITHMETIC

Lecture 10: Strongly definable types

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> There is a paradoxical link with *stable* first-order theories: although the notion of definable types was introduced by Gaifman in the study of PA, which is the most unstable theory, this notion turned out to be a fundamental one for stable theories. [...] I expect (i) that it will not be possible to "explain" this similarity by a (reasonable) common mathematical theory; and (ii) that this similarity is *not* superficial, however. Although they cannot be "captured" mathematically, such similarities do occur repeatedly and not by chance in the development of two opposite parts of logic, namely, model theory of algebraic style on the one hand, and the theory (model, proof, recursion, and set theory) of the basic *universes* (e.g. arithmetic, analysis, V, etc.) and their axiomatic systems on the other.
>
> Jean-Pierre Ressayre [5]

Let us carry on our investigation about types. We are still focusing on end extensions. Hence we will restrict our attention to types whose realizations are on top of all old elements.

Definition. A type $p(v)$ over an $\mathscr{L}_{\mathrm{A}}$-structure M is *unbounded* if $p(v) \supseteq \{v > a : a \in M\}$.

If $K \succcurlyeq M \models \mathrm{PA}^-$ and K realizes an unbounded type over M, then $K \not\supseteq_{\mathrm{cf}} M$ and so $K \neq M$.

Notation. Write $\exists^\infty v\ \theta(v)$ for $\forall v_0\ \exists v \geqslant v_0\ \theta(v)$, meaning there are unboundedly (or cofinally) many v satisfying θ. The dual notation $\forall^\infty v\ \theta(v)$ stands for $\neg\exists^\infty v\ \neg\theta(v)$, meaning all but boundedly many v satisfy θ, or equivalently, eventually every v satisfies θ.

Let $M \models \mathrm{PA}^-$ and $p(v)$ be an unbounded complete M-type. Notice unboundedness implies $M \models \exists^\infty v\ \theta(v)$ for every $\theta(v) \in p(v)$. Given $\varphi(v) \in \mathscr{L}_{\mathrm{A}}(M)$, if we can find $\theta(v) \in p(v)$ such that

$$M \models \forall^\infty v\ \big(\theta(v) \rightarrow \varphi(v)\big),$$

then $\neg\varphi(v) \notin p(v)$ because $M \models \neg\exists^\infty v\ \big(\theta(v) \wedge \neg\varphi(v)\big)$, and so $\varphi(v) \in p(v)$ by the completeness of $p(v)$. In this case, we can think of $\theta(v)$ as *forcing* $\varphi(v)$ into $p(v)$. The formula $\theta(v) \in p(v)$ *decides* $\varphi(v) \in \mathscr{L}_{\mathrm{A}}(M)$ if $\theta(v)$ forces either $\varphi(v)$ or $\neg\varphi(v)$ into $p(v)$. Since p is a complete M-type, every $\mathscr{L}_{\mathrm{A}}(M)$-formula is decided by some element of $p(v)$, to wit, either the formula itself or its negation. In general, an infinite family of $\mathscr{L}_{\mathrm{A}}(M)$-formulas needs infinitely many formulas to decide. However, if the family is uniformly definable, then it is possible to decide the whole family simultaneously by a single formula.

Definition. Let $M \models \mathrm{PA}^-$. A complete M-type $p(v)$ is *strongly definable* if for every $\varphi(v, z) \in \mathscr{L}_{\mathrm{A}}$, there exists $\theta(v) \in p(v)$ such that

$$M \models \forall z\ \Big(\forall^\infty v\ \big(\theta(v) \rightarrow \varphi(v, z)\big) \vee \forall^\infty v\ \big(\theta(v) \rightarrow \neg\varphi(v, z)\big)\Big). \qquad (*)$$

Strongly definable types are also called *end-extensional types* in the literature.

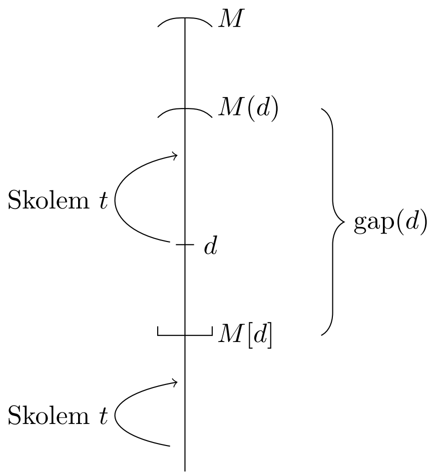

Figure 10.1: The gap in M containing d

Substituting $S = \{v \in M : M \models \theta(v)\}$ and $R = \{\langle z, v\rangle \in M : M \models \varphi(v, z)\}$ into $(*)$ gives

$$M \models \forall z \left(S \subseteq^* (R)_z \vee S \subseteq^* (R)_z^{\mathrm{c}}\right),$$

which is reminiscent of COH from the previous lecture. With this in mind, we may disassemble the proof of Theorem 9.10 into two separate parts.

Theorem 10.1 (Gaifman [2])**.** Every $M \models \mathrm{PA}$ admits an unbounded strongly definable type. □

Lemma 10.2. Let $M \models \mathrm{PA}^-$. Then every unbounded strongly definable complete M-type is definable. □

It follows that extensions of a model of PA by an unbounded strongly definable type is a proper elementary end extension by Propositions 9.5 and 9.6. There are, however, unbounded definable types which fail to be strongly definable [2, Proposition 2.23].

10.1 Gaps

In this section, we investigate how our elementary end extensions 'look like' over the ground model. One useful way to 'visualize' models of PA is via *gaps*.

Definition. A *Skolem function* or *term* is a parameter-free definable PA-provably total function. Let $d \in M \models \mathrm{PA}$. Then

- $M(d) = \{x \in M : x < t(d) \text{ for some Skolem function } t\}$;
- $M[d] = \{x \in M : t(x) < d \text{ for every Skolem function } t\}$;
- the *gap* (or *sky*) in M containing d, denoted $\mathrm{gap}(d)$, is $M(d) \setminus M[d]$.

Skolem functions are typically of the form

$$v \mapsto (\min u)(\eta(u, v)) \quad \text{or} \quad v \mapsto (\max u)(\eta(u, v)),$$

where $\eta \in \mathscr{L}_{\mathrm{A}}$. Unless otherwise stated, these extrema are assumed to take value 0 by convention when they fail to exist. Let $d \in M \models \mathrm{PA}$. Then it can be shown that $M(d)$ is the smallest elementary cut of M containing d. The set $M[d]$ is empty if and only if every elementary cut of M contains d. Notice if $x < y \in M[d]$, then for every Skolem function t,

$$t(x) \leqslant \max\{t(y') : y' \leqslant y\} < d.$$

So $M[d]$ is an initial segment of M. Moreover, if $M[d] \neq \varnothing$, then it is the biggest elementary cut of M which does not contain d. It follows that $\mathrm{gap}(d)$ is always a convex subset of M containing d.

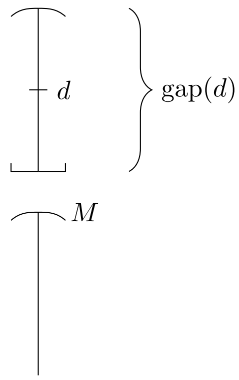

Figure 10.2: Extension of $M \models \mathrm{PA}$ by d realizing an unbounded strongly definable type

Lemma 10.3. Let $c, d \in M \models \mathrm{PA}$. Then $c \in \operatorname{gap}(d)$ if and only if there are Skolem functions t, s such that

$$t(c) \geqslant d \quad \text{and} \quad s(d) \geqslant c.$$

Proof. These conditions say respectively $c > M[d]$ and $c \in M(d)$. □

We will look at all the possible configurations of new gaps in elementary end extensions in general in the next lecture. For now, let us start with the simplest picture. It is given by unbounded strongly definable types.

Definition. Let $M \models \mathrm{PA}$ and $p(v)$ be a complete M-type. Then we write $M(p/d)$ for the $\mathscr{L}_{\mathrm{A}}$-reduct of the prime model of $p(d)$ as given by Proposition 9.3, where d is a new constant symbol, and we also use d to refer to the interpretation of the symbol d in this prime model.

Proposition 10.4 (Gaifman [2, Proposition 4.8])**.** Let $M \models \mathrm{PA}$ and $p(v)$ be an unbounded strongly definable complete M-type. Then $M(p/d)$ is the disjoint union of M and $\operatorname{gap}(d)$.

Proof. Since $p(v)$ is unbounded, we know $d > M$. So $t(x) \in M < d$ for every $x \in M$ and every Skolem function t. This implies $M \subseteq M(p)[d]$ and thus $M \cap \operatorname{gap}(d) = \varnothing$.

Take $c \in M(p)$. We will show $c \in M$ or $c \in \operatorname{gap}(d)$. Find $a \in M$ and $\eta \in \mathscr{L}_{\mathrm{A}}$ such that $c = (\min x)(\eta(x, a, d))$. Replacing $\eta(x, z, v)$ with the formula $x = (\min x')(\eta(x', z, v))$ if necessary, we may assume

$$M(p) \models \forall v, z\ \exists! x\ \eta(x, z, v). \tag{†}$$

Using strong definability, find $\theta(v, b) \in p(v)$, where $\theta \in \mathscr{L}_{\mathrm{A}}$ and $b \in M$, such that

$$M \models \forall x, z\ \Big(\forall^\infty v\ \big(\theta(v, b) \to \eta(x, z, v)\big) \vee \forall^\infty v\ \big(\theta(v, b) \to \neg\eta(x, z, v)\big)\Big).$$

Case 1. Suppose $M \models \forall x\ \forall^\infty v\ \big(\theta(v, b) \to \neg\eta(x, z, v)\big)$. If $c \in M$, then we are done already. So suppose not. Then $c > M$ by Lemma 10.2, Proposition 9.5, and Proposition 9.6. Transferring our assumption to $M(p)$, we get

$$t(c) = \max\big\{(\max v)\big(\theta(v, w) \wedge \eta(c, z, v)\big) : w, z \leqslant c\big\} \geqslant d$$

on the one hand, because $a, b \in M < c$ and $M \models \eta(c, a, d)$. On the other hand,

$$s(d) = \max\big\{(\min x)\big(\eta(x, z, d)\big) : z \leqslant d\big\} \geqslant c,$$

because unboundedness implies $a \in M < d$. Therefore $c \in \operatorname{gap}(d)$ by Lemma 10.3.

Case 2. Suppose $M \models \exists x\ \exists^\infty v\ \big(\theta(v, b) \wedge \eta(x, z, v)\big)$. Let $a_0 \in M$ witness this. Then $M \models \forall^\infty v\ \big(\theta(v, b) \to \eta(a_0, a, v)\big)$ by the choice of θ. Thus for some fixed $v_0 \in M$,

$$M \models \forall v\ \big(\theta(v, b) \wedge v \geqslant v_0 \to \eta(a_0, a, v)\big).$$

This transfers up to $M(p)$. So $M(p) \models \eta(a_0, a, d)$ since $v_0 \in M < d$ and $M(p) \models p(d) \ni \theta(d, b)$. Therefore $c = a_0 \in M$ by (†). □

10.2 Extensions of types

One of the main features of definable types is that they can be extended canonically via their defining schemes. More precisely, suppose M is a structure for a language $\mathscr{L}$, and $p(\bar{v})$ is a complete M-type. Let $K \succcurlyeq M$. Then of course there is a complete K-type $q(\bar{v}) \supseteq p(\bar{v})$ because

$$K \succcurlyeq M \models \exists \bar{v}\ \theta(\bar{v})$$

for every $\theta(\bar{v}) \in p(\bar{v})$. Such $q(\bar{v})$ is called an *extension* of $p(\bar{v})$. If $p(\bar{v})$ is a definable type, then we can pick out a canonical extension by setting for all $\bar{z} \in K$,

$$\varphi(\bar{v}, \bar{z}) \in q(\bar{v}) \quad \Leftrightarrow \quad K \models \chi(\bar{z}),$$

where $\chi \in \mathscr{L}_{\mathrm{A}}(M)$ such that $\{\bar{z} \in M : \varphi(\bar{v}, \bar{z}) \in p(\bar{v})\} = \{\bar{z} \in M : M \models \chi(\bar{z})\}$. This enables us to extend a given structure repeatedly using the same definable type.

Notice in the context of arithmetic, extensions of unbounded types can always be made unbounded, because $K \succcurlyeq M \models \exists^\infty v\ \theta(v)$ for all $\theta(v) \in p(v)$ when p is unbounded. Complete extensions of strongly definable types are always strongly definable because the sentences in $(*)$ are preserved in all elementary extensions. Those sentences actually show that strongly definable types extend uniquely. Moreover, the converse is true.

Theorem 10.5 (Gaifman [2, Theorem 2.10])**.** Let $M \models \mathrm{PA}^-$. Then the following are equivalent for an unbounded complete M-type $p(v)$.

(a) $p(v)$ is strongly definable.

(b) For every $K \succcurlyeq M$, there is at most one unbounded complete K-type $q(v) \supseteq p(v)$.

Proof. First consider (a) $\Rightarrow$ (b). Suppose $p(v)$ is strongly definable. Let $K \succcurlyeq M$ and $q_1(v), q_2(v)$ be unbounded complete K-types extending $p(v)$. Pick any $\varphi(v, z) \in \mathscr{L}_{\mathrm{A}}$ and $a \in K$ such that $\varphi(v, a) \in q_1(v)$. We will then show $\varphi(v, a) \in q_2(v)$. This suffices for (b) by symmetry. Apply the strong definability of p to find $\theta(v) \in p(v)$ such that

$$K \succcurlyeq M \models \forall z \left(\forall^\infty v\ \big(\theta(v) \to \varphi(v, z)\big) \vee \forall^\infty v\ \big(\theta(v) \to \neg\varphi(v, z)\big)\right).$$

Since $\theta(v) \in p(v) \subseteq q_1(v)$ and $\varphi(v, a) \in q_1(v)$, we know $K \models \exists^\infty v\ \big(\theta(v) \wedge \varphi(v, a)\big)$ because $q_1(v)$ is unbounded. Thus $K \models \forall^\infty v\ \big(\theta(v) \to \varphi(v, a)\big)$ by the choice of θ. This implies $\neg\varphi(v, a) \notin q_2(v)$ because $K \models \neg\exists^\infty v\ \big(\theta(v) \wedge \neg\varphi(v, a)\big)$ and $q_2(v)$ is unbounded. So, as $q_2(v)$ is complete, we conclude $\varphi(v, a) \in q_2(v)$, as required.

Next, consider the implication (b) $\Rightarrow$ (a). We will show that either (b) fails or (a) holds. Let

$$r(z) = \left\{\exists^\infty v\ \big(\theta(v) \wedge \varphi(v, z)\big) \wedge \exists^\infty v\ \big(\theta(v) \wedge \neg\varphi(v, z)\big) : \theta(v) \in p(v)\right\}.$$

Case 1. Suppose $\mathrm{ElemDiag}(M) + r(c)$ is consistent, where c is a new constant symbol. Elementarily extend M to $K \models r(c)$. In this model K, both

$$p(v) \cup \{v > a : a \in K\} \cup \{\varphi(v, c)\} \quad \text{and} \quad p(v) \cup \{v > a : a \in K\} \cup \{\neg\varphi(v, c)\}$$

are finitely satisfied. So they extend to different unbounded complete K-types, making (b) fail.

Case 2. Suppose $\mathrm{ElemDiag}(M) + r(c)$ is not consistent. Use the Compactness Theorem to find $\theta_0(v), \theta_1(v), \ldots, \theta_n(v) \in p(v)$ such that

$$M \models \forall z \left(\bigvee_{i \leqslant n} \forall^\infty v\ \big(\theta_i(v) \to \varphi(v, z)\big) \vee \bigvee_{i \leqslant n} \forall^\infty v\ \big(\theta_i(v) \to \neg\varphi(v, z)\big)\right). \tag{‡}$$

Define $\theta(v) = \bigwedge_{i \leqslant n} \theta_i(v)$. Then $\theta(v) \in p(v)$ too because $p(v)$ is complete. We claim that

$$M \models \forall z\ \big(\forall^\infty v\ (\theta(v) \to \varphi(v, z)) \vee \forall^\infty v\ (\theta(v) \to \neg\varphi(v, z))\big),$$

which is what we need for (a). Take any $z \in M$. Then at least one of the $2n + 2$ disjuncts in (‡) is true. If $i \leqslant n$ such that $M \models \forall^\infty v\ \big(\theta_i(v) \to \varphi(v, z)\big)$, then $M \models \forall^\infty v\ \big(\theta(v) \to \varphi(v, z)\big)$ because $M \models \forall v\ \big(\theta(v) \to \theta_i(v)\big)$. Similarly, if $i \leqslant n$ such that $M \models \forall^\infty v\ \big(\theta_i(v) \to \neg\varphi(v, z)\big)$, then $M \models \forall^\infty v\ \big(\theta(v) \to \neg\varphi(v, z)\big)$. So the claim is proved. ☐

Further exercises

Let us consider the following converse to Theorem 10.5.

Theorem 10.6 (Gaifman [2, Theorem 2.21])**.** Let $M \models \mathrm{PA}$ and $p(v)$ be a complete M-type. Then the following are equivalent.

(a) $p(v)$ is unbounded and strongly definable.

(b) For every $K \succcurlyeq M$ and every complete K-type $q(v) \supseteq p(v)$, we have $K(q/b) = K \cup \operatorname{gap}(b)$.

Proof. We almost proved one direction already.

(1) Show that (a) $\Rightarrow$ (b).

(2) Show that if (b) holds, then $p(v)$ is unbounded.

For the rest of the proof, we follow the argument for (1) $\Rightarrow$ (4) in Theorem 2.10 of Gaifman [2]. Suppose (b) holds. In view of Theorem 10.5, it suffices to show that whenever $q(v), q'(v)$ are complete K-types extending $p(v)$, where $K \succcurlyeq M$, we have $q = q'$. Let $K(q) = K(q/d)$ and $K(q, q^*) = K(q)(q^*/d^*)$, where $q^*(v)$ is some complete $K(q)$-type extending $q'(v)$. Then by (b),

$$K(q, q^*) = K \cup \operatorname{gap}_{K(q,q^*)}(d) \cup \operatorname{gap}_{K(q,q^*)}(d^*). \qquad (\S)$$

Fix $\varphi(v, z) \in \mathscr{L}_\mathrm{A}$. We claim that $\varphi(v, z) \in q(v)$ if and only if $\varphi(v, z) \in q'(v)$ for every $z \in K$. If $K(q, q^*) \models \forall z\ \big(\varphi(d, z) \leftrightarrow \varphi(d^*, z)\big)$, then we are done. So suppose not. Changing φ to $\neg\varphi$ if necessary, we may assume $K(q, q^*) \models \exists z\ \big(\varphi(d, z) \wedge \neg\varphi(d^*, z)\big)$. Define

$$c = (\min z)\big(\varphi(d, z) \wedge \neg\varphi(d^*, z)\big)$$

within $K(q, q^*)$. It suffices to show $c > K$. Let $M(p) = M(p/d)$ and

$$p^*(v) = \big\{\theta(v) \in \mathscr{L}_\mathrm{A}(M(p)) : \theta(v) \in q^*(v)\big\}.$$

Set $M(p, p^*) = M(p)(p^*/d^*)$.

(3) Explain why we can identify these new d, d^* with the old ones and view $M(p, p^*) \preccurlyeq K(q, q^*)$.

(4) Explain why $c \in M(p, p^*)$.

(5) Recall $q(v), q'(v)$ both extend the complete M-type $p(v)$. Show $c \notin M$.

(6) Explain why $M(p, p^*) = M \cup \operatorname{gap}_{M(p,p^*)}(d) \cup \operatorname{gap}_{M(p,p^*)}(d^*)$.

(7) Put everything together, and show $c > K$. □

Further comments

Overloading $M(d)$

Let $M \models \mathrm{PA}$. Recall $M(d)$ denotes the smallest elementary cut of M containing d. In the literature, sometimes $M(d)$ is used to denote the model of PA obtained from M by adjoining an element d. We do not use this notation here. Unfortunately, the two meanings of $M(d)$ are not always the same.

To see this, fix any countable nonstandard $M \models \mathrm{Th}(\mathbb{N})$. Pick any $d \in M \setminus \mathbb{N}$. Use the Compactness Theorem to find $K \succcurlyeq M$ with a new element c between $\mathbb{N}$ and $M \setminus \mathbb{N}$. Then $c \in K(d)$ because $c < d \in K(d) \subseteq_\mathrm{e} K$. Let us denote by $\mathrm{cl}(d)$ the smallest elementary substructure of K that contains d (and all elements of $\mathbb{N}$). Then $\mathrm{cl}(d)$ is the smallest elementary extension of $\mathbb{N}$ that contains d, so that $\mathrm{cl}(d) \subseteq M$. Hence $K(d) \neq \mathrm{cl}(d)$ since $c \notin M$.

Nevertheless, if $M \models \mathrm{PA}$ and $p(v)$ is an unbounded complete M-type, then $M(p) = M(p/d)$ implies $M(p)(d) = M(p)$.

Further reading

Model-theoretic forcing

What we have been doing in these two lectures can be viewed as a kind of model-theoretic forcing. Let us make this more explicit here. Consider $M \models \mathrm{PA}$. The poset we are working in is

$$\mathrm{Def}^*(M) = \{S \in \mathrm{Def}(M) : S \subseteq_{\mathrm{cf}} M\}$$

with inclusion. As usual, a subset $\mathscr{D} \subseteq \mathrm{Def}^*(M)$ is *dense* if

$$\forall S{\in}\mathrm{Def}^*(M)\ \ \exists S'{\in}\mathscr{D}\ \ S' \subseteq S.$$

The fact that $(M, \mathrm{Def}(M)) \models \mathrm{COH}$ says

$$\mathscr{D}_\varphi = \big\{H \in \mathrm{Def}^*(M) : M \models \forall z\ \big(\forall^\infty v{\in}H\ \varphi(v,z) \vee \forall^\infty v{\in}H\ \neg\varphi(v,z)\big)\big\}$$

is dense for every $\varphi(v,z) \in \mathscr{L}_\mathrm{A}$.

A *filter* is a nonempty family of sets from $\mathrm{Def}^*(M)$ that are closed under taking supersets and finite intersections. A filter $\mathscr{F}$ is said to *meet* $\mathscr{D} \subseteq \mathrm{Def}^*(M)$ if $\mathscr{F} \cap \mathscr{D} \neq \varnothing$. The proof of Theorem 9.10 shows any countably many dense sets can be met simultaneously by some filter.

A filter is *arithmetically generic* if it meets all dense sets that are parameter-free arithmetically definable in $(M, \mathrm{Def}(M))$. Every filter $\mathscr{F}$ corresponds to a type

$$p(v) = \{\theta(v) : \theta \text{ defines some } S \in \mathscr{F}\}$$

over M. Let $\mathscr{G}$ be an arithmetically generic filter. In view of the $\mathscr{D}_\varphi$'s, the type $p(v)$ corresponding to $\mathscr{G}$ is an unbounded strongly definable complete M-type. So we can define the *arithmetically generic* extension $M[\mathscr{G}] = M(p)$. One may also write $M[\mathscr{G}/d] = M(p/d)$. Notice every element of $M[\mathscr{G}/d]$ has a name $(\min x)(\eta(x,d))$, where $\eta \in \mathscr{L}_\mathrm{A}(M)$, as given by Proposition 9.3.

Let $S \in \mathrm{Def}^*(M)$ and $\varphi(v) \in \mathscr{L}_\mathrm{A}(M)$. Then we say S *forces* $\varphi(v)$, and write $S \Vdash \varphi(v)$, if $M[\mathscr{G}/d] \models \varphi(d)$ for every arithmetically generic filter $\mathscr{G}$ containing S. Equivalently $S \Vdash \varphi(v)$ if and only if $M \models \forall^\infty v{\in}S\ \varphi(v)$, as mentioned in the introduction.

In the present lecture, we looked at some basic properties of arithmetically generic filters and arithmetically generic extensions. We will see a little more of this in the next lecture.

Many arguments in model theory have a hint of forcing in them. We have already met some in Lectures 5, 7 and 8. For a general introduction, see Hodges's book [3]. All this discussion about types can also be translated to the language of (ultra)filters. For example, strongly definable types correspond to P-points [1], while rare types, which we will meet in the next lecture, correspond to Q-points. See Kirby's article [4] for more details about this translation.

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