MODEL THEORY OF ARITHMETIC

Lecture 15: Forcing in arithmetic

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> Cohen taught that truth can be approximated more easily and more completely in the intermediate stages of a construction than any recursion theorist would have believed possible [...].
>
> Gerald E. Sacks [10, Introduction]

The aim of this lecture is to show a conservation theorem between RCA_0 and WKL_0.

Theorem 15.1 (Harrington [11, Corollary IX.2.6], Ratajczyk [9], independently)**.** WKL_0 is Π^1_1-conservative over RCA_0, i.e., for all sentences $\sigma \in \Pi^1_1$, if $\mathrm{WKL}_0 \vdash \sigma$, then $\mathrm{RCA}_0 \vdash \sigma$.

Remark 15.2. WKL_0 is not Σ^1_1-conservative over RCA_0, because there is an infinite recursive binary tree with no infinite recursive branch, cf. Corollary 4.9. Nevertheless, Theorem 15.1 can still be improved: Simpson–Tanaka–Yamazaki [13] showed that WKL_0 is conservative over RCA_0 for all sentences of the form $\forall X\ \exists! Y\ \theta(X, Y)$, where θ is arithmetical.

15.1 Adding second-order objects

There are many ways in which one can prove Theorem 15.1. Here let us follow a model-theoretic approach similar to that we used in Lecture 6 for the conservation result between $\mathrm{I}\Sigma_n$ and $\mathrm{B}\Sigma_{n+1}$. An $\mathscr{L}_{\mathrm{II}}$-structure is *countable* if both its first- and second-order parts are countable.

Theorem 15.3 (Harrington [11, Theorem IX.2.1])**.** For every countable $(M, \mathscr{X}) \models \mathrm{RCA}_0$, there exists $\mathscr{Y} \supseteq \mathscr{X}$ such that $(M, \mathscr{Y}) \models \mathrm{WKL}_0$.

Similar to how Theorem 6.6 implies Theorem 6.1, one can quickly show Theorem 15.1 from Theorem 15.3.

Proof of Theorem 15.1. Let $\theta(\bar{X})$ be an arithmetical formula such that $\mathrm{RCA}_0 + \exists \bar{X}\ \theta(\bar{X})$ is consistent. Take a countable $(M, \mathscr{X}) \models \mathrm{RCA}_0 + \theta(\bar{A})$ where $\bar{A} \in \mathscr{X}$. Apply Theorem 15.3 to get $\mathscr{Y} \supseteq \mathscr{X}$ such that $(M, \mathscr{Y}) \models \mathrm{WKL}_0$. Notice $\theta(\bar{X})$ involves only first-order quantifiers. So, as $(M, \mathscr{X})$ and $(M, \mathscr{Y})$ have the same first-order part, we know $(M, \mathscr{Y}) \models \theta(\bar{A})$ too. Therefore $\mathrm{WKL}_0 + \exists \bar{X}\ \theta(\bar{X})$ is consistent. □

Recall Example 4.4(2), which tells us that the $\mathscr{L}_{\mathrm{A}}$-consequences of RCA_0 is axiomatized by $\mathrm{I}\Sigma_1$. We present a slightly more general statement here. If $(M, \mathscr{X})$ is an $\mathscr{L}_{\mathrm{II}}$-structure, then Δ^0_1-$\mathrm{Def}(M, \mathscr{X})$ denotes the set of all parametrically Δ^0_1-definable subsets of M in $(M, \mathscr{X})$. Notice in Δ^0_1-$\mathrm{Def}(M, \mathscr{X})$ we do not include any Δ^0_1-definable subset of $\mathscr{X}$ in $(M, \mathscr{X})$.

Lemma 15.4. Let $(M, \mathscr{X}_0) \models \mathrm{I}\Sigma^0_1$. Then

$$(M, \mathscr{X}) = (M, \Delta^0_1\text{-}\mathrm{Def}(M, \mathscr{X}_0)) \models \mathrm{RCA}_0.$$

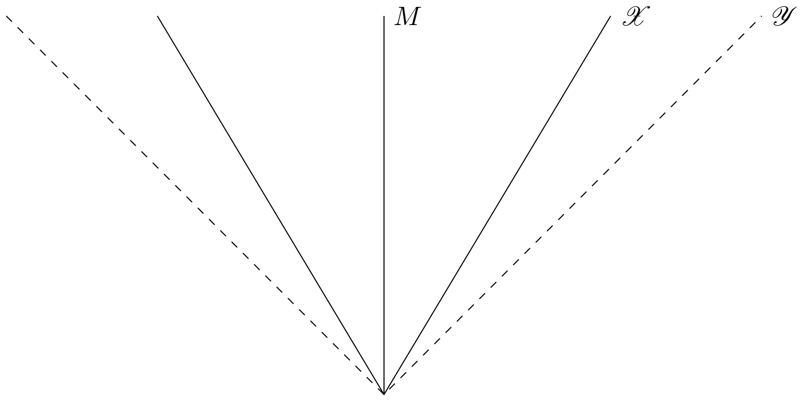

Figure 15.1: Extending $(M, \mathscr{X}) \models \mathrm{RCA}_0$ to $(M, \mathscr{Y}) \models \mathrm{WKL}_0$

Proof. Notice if $\alpha(\bar{z})$ is an atomic or negated atomic $\mathscr{L}_{\mathrm{II}}(M, \mathscr{X})$-formula, then we can replace a parameter from $\mathscr{X}$ with its Σ^0_1 or Π^0_1 definition to obtain a $\Sigma^0_1(M, \mathscr{X}_0)$-formula $\alpha_0(\bar{z})$ such that

$$(M, \mathscr{X}) \models \alpha(\bar{c}) \quad \Leftrightarrow \quad (M, \mathscr{X}_0) \models \alpha_0(\bar{c})$$

for all $\bar{c} \in M$. Moreover, every Σ^0_1-formula is logically equivalent to one in *prenex form*, i.e., one in which negations are only applied to atomic formulas, and no unbounded quantifier appears inside the scope of a bounded quantifier. Hence for every $\Sigma^0_1(M, \mathscr{X})$-formula $\theta(\bar{z})$, there exists an $\mathscr{L}_{\mathrm{II}}(M, \mathscr{X}_0)$-formula $\theta_0(\bar{z})$ of the form

$$\underbrace{\exists \bar{x}\ \forall \bar{y}_1{<}t_1\ \exists \bar{y}_2{<}t_2\ \cdots \bigwedge_i \bigvee_j \underbrace{\alpha_{ij}(\bar{x}, \bar{y}, \bar{z})}_{\Sigma^0_1(M, \mathscr{X}_0)}}_{\Sigma^0_1(M, \mathscr{X}_0) \text{ over } \mathrm{B}\Sigma^0_1},$$

where each $\alpha_{ij} \in \Sigma^0_1(M, \mathscr{X}_0)$, such that for every $\bar{c} \in M$,

$$(M, \mathscr{X}) \models \theta(\bar{c}) \quad \Leftrightarrow \quad (M, \mathscr{X}_0) \models \theta_0(\bar{c}).$$

Hence $\mathrm{I}\Sigma^0_1$ in $(M, \mathscr{X}_0)$ gives us $\mathrm{I}\Sigma^0_1$ in $(M, \mathscr{X})$. Similarly $\Delta^0_1\text{-Def}(M, \mathscr{X}) \subseteq \Delta^0_1\text{-Def}(M, \mathscr{X}_0) = \mathscr{X}$, so that $(M, \mathscr{X}) \models \Delta^0_1\text{-CA}$. □

As a consequence, if we want an extension satisfying RCA_0, then it suffices to obtain an extension satisfying $\mathrm{I}\Sigma^0_1$, because after that we can close up under Δ^0_1-comprehension to get a model of RCA_0. Moreover, if $(M, \mathscr{X}_0)$ is countable, then so is $(M, \Delta^0_1\text{-Def}(M, \mathscr{X}_0))$. With this, we can further reduce Theorem 15.3 to a simpler proposition, which says essentially that we can always add an unbounded branch to an unbounded tree while preserving $\mathrm{I}\Sigma^0_1$.

Proposition 15.5 (Harrington [11, Lemma IX.2.5])**.** Take any countable $(M, \mathscr{X}) \models \mathrm{I}\Sigma^0_1$. Let T be an unbounded binary tree in $\mathscr{X}$. Then there is an unbounded branch $B \subseteq T$ such that $(M, \mathscr{X} \cup \{B\}) \models \mathrm{I}\Sigma^0_1$.

After adding a new branch using Proposition 15.5, one may get more trees, but then we can apply Proposition 15.5 again to get another branch through another tree, etc. Since all the models are countable, after ω-many steps, the addition of branches catches up with the increase in trees, so that every unbounded tree gets an unbounded branch at the end. This gives us the Weak König Lemma, as required by Theorem 15.3.

Proof of Theorem 15.3. Suppose we are given a countable $(M, \mathscr{X}) \models \mathrm{RCA}_0$. Using Proposition 15.5, build, by recursion, countable sets

$$\mathscr{X} = \mathscr{X}_0 \subseteq \mathscr{X}_1 \subseteq \mathscr{X}_2 \subseteq \cdots$$

such that for each $i \in \mathbb{N}$,

- $(M, \mathscr{X}_i) \models \mathrm{RCA}_0$; and
- if T is an unbounded tree in $\mathscr{X}_i$, then there are $j \in \mathbb{N}$ and $B \in \mathscr{X}_j$ such that B is an unbounded branch in T.

Then $(M, \mathscr{Y}) = \big(M, \bigcup_{i \in \mathbb{N}} \mathscr{X}_i\big) \models \mathrm{WKL}_0$, because each instance of $\mathrm{I}\Sigma^0_1$ and Δ^0_1-CA is only about finitely many elements in $\mathscr{Y}$, and so it is determined already at a finite stage of the construction. □

The Weak König Lemma and comprehension schemes such as Δ^0_1-CA are often called *set existence axioms* because they assert the closure of the second-order universe under certain operations. In contrast, induction schemes such as $\mathrm{I}\Sigma^0_1$ are properties of numbers in which second-order objects only act as parameters. This is why we can add *one* set to a model of second-order arithmetic while preserving $\mathrm{I}\Sigma^0_1$, but adding one (new) set while preserving Δ^0_1-CA is impossible.

15.2 Kleene Normal Form

To prove Proposition 15.5, we need a better understanding of the Σ^0_1-formulas. Recall from Lecture 4 that Σ^0_1-formulas can be viewed as programs, in which set parameters act as oracles. Intuitively, if P is a program with oracle $A \subseteq \mathbb{N}$ and P halts on an input $\bar{x} \in \mathbb{N}$, then a run of P on input $\bar{x}$ can only call $\mathrm{Prog}\langle x \in A\rangle$ finitely many times, and so only a finite part of A is relevant for this particular computation. The next theorem is a mathematical way of putting this. For the rest of this lecture, we identify a set S with its characteristic function χ_S. Notice S and χ_S are mutually Δ^0_0-definable. Now $S{\restriction}a$ denotes the usual restriction of the function S to a, so that it carries, in addition to positive information, also negative information about the set S, unlike our definition in Lecture 2. With $\mathrm{I}\Delta^0_0 + \exp$, we can code $S{\restriction}a$ by a first-order object using a second-order version of Theorem 2.7. Without $\mathrm{I}\Delta^0_0 + \exp$, we can read $v \in S{\restriction}a$ and $v \notin S{\restriction}a$ respectively as

$$v < a \wedge v \in S \quad \text{and} \quad v < a \wedge v \notin S,$$

and set $\mathrm{len}(S{\restriction}a) = a$. These are the only ways in which $S{\restriction}a$ will appear in our formulas.

Definition. A Σ^0_1-formula is in *Kleene Normal Form* if it is

$$\exists \ell\ \eta(\bar{m}, X{\restriction}\ell)$$

for some $\eta \in \Delta^0_0$, possibly with undisplayed free variables, such that

$$\mathrm{PA}^- \vdash \forall X\ \forall \ell, \bar{m}\ \big(\eta(\bar{m}, X{\restriction}\ell) \rightarrow \forall \ell' {\geqslant} \ell\ \eta(\bar{m}, X{\restriction}\ell')\big). \tag{$*$}$$

The set of all Σ^0_1-formulas in Kleene Normal Form is denoted $\Sigma^0_1{\restriction}\mathrm{KNF}$.

The monotonicity property $(*)$ can be paraphrased as saying that the acquirement of (non-contradictory) new information does not change a computation.

Kleene Normal Form Theorem for Σ^0_1-formulas [7, pages 290–292]. Every Σ^0_1-formula $\theta(\bar{m}, X)$ is uniformly equivalent to one in Kleene Normal Form over $\mathrm{PA}^- + \mathrm{Coll}(\Sigma^0_1{\restriction}\mathrm{KNF})$.

Proof sketch. We proceed by an induction on θ in prenex form, cf. the proof of Lemma 15.4.

If $\theta(\bar{m}, X)$ is an atomic formula not involving X, then we can let $\eta = \theta$. If $\theta(\bar{m}, X)$ is $t(\bar{m}) \in X$, where t is an $\mathscr{L}_{\mathrm{A}}$-term, then define $\eta(\bar{m}, X{\restriction}\ell)$ to be $t(\bar{m}) \in X{\restriction}\ell$. Similarly, if $\theta(\bar{m}, X)$ is $t(\bar{m}) \notin X$, then we may let $\eta(\bar{m}, X{\restriction}\ell)$ be $t(\bar{m}) \notin X{\restriction}\ell$. Condition $(*)$ can easily be verified in view of the comments made at the beginning of this section.

Suppose $\theta(\bar{m}, X)$ is $\theta_0(\bar{m}, X) \vee \theta_1(\bar{m}, X)$. Using the induction hypothesis, find Σ^0_1-formulas $\exists \ell_0\ \eta_0(\bar{m}, X{\restriction}\ell_0)$ and $\exists \ell_1\ \eta_1(\bar{m}, X{\restriction}\ell_1)$ in Kleene Normal Form that are uniformly equivalent to $\theta_0(\bar{m}, X)$ and $\theta_1(\bar{m}, X)$ in $\mathrm{PA}^- + \mathrm{Coll}(\Sigma^0_1{\restriction}\mathrm{KNF})$ respectively. Then we can set $\eta(\bar{m}, X{\restriction}\ell)$ to be

$$\eta_0(\bar{m}, X{\restriction}\ell) \vee \eta_1(\bar{m}, X{\restriction}\ell).$$

The case when $\theta(\bar{m}, X)$ is $\theta_0(\bar{m}, X) \wedge \theta_1(\bar{m}, X)$ is dealt with in essentially the same way.

The case for bounded existential quantification is similar to that for bounded universal quantification. So suppose $\theta(\bar{m}, X)$ is $\forall m'{<}t\ \theta_0(\bar{m}, m', X)$, where t is an $\mathscr{L}_\mathrm{A}$-term. Using the induction hypothesis, find a Σ^0_1-formula $\exists \ell_0\ \eta_0(\bar{m}, m', X{\restriction}\ell_0)$ in Kleene Normal Form that is uniformly equivalent to $\theta_0(\bar{m}, m', X)$ over $\mathrm{PA}^- + \mathrm{Coll}(\Sigma^0_1{\restriction}\mathrm{KNF})$. Then the following equivalences hold in $\mathrm{PA}^- + \mathrm{Coll}(\Sigma^0_1{\restriction}\mathrm{KNF})$.

- $$\theta(\bar{m}, X) \leftrightarrow \forall m'{<}t\ \exists \ell_0\ \eta_0(\bar{m}, m', X{\restriction}\ell_0).$$
- $$\forall m'{<}t\ \exists \ell_0\ \eta_0(\bar{m}, m', X{\restriction}\ell_0) \leftrightarrow \exists \ell\ \forall m'{<}t\ \exists \ell_0{\leqslant}\ell\ \eta_0(\bar{m}, m', X{\restriction}\ell_0) \qquad \text{by } \mathrm{Coll}(\Sigma^0_1{\restriction}\mathrm{KNF}).$$
- $$\exists \ell\ \forall m'{<}t\ \exists \ell_0{\leqslant}\ell\ \eta_0(\bar{m}, m', X{\restriction}\ell_0) \leftrightarrow \exists \ell\ \forall m'{<}t\ \eta_0(\bar{m}, m', X{\restriction}\ell) \qquad \text{by } (*) \text{ for } \eta_0.$$

One can then set $\eta(\bar{m}, X{\restriction}\ell)$ to be $\forall m'{<}t\ \eta_0(\bar{m}, m', X{\restriction}\ell)$. We know η again satisfies property $(*)$ because η_0 appears only positively in η.

Finally, suppose $\theta(\bar{m}, X)$ is $\exists m'\ \theta_0(\bar{m}, m', X)$. If $\exists \ell_0\ \eta_0(\bar{m}, m', X{\restriction}\ell_0)$ is a Σ^0_1-formula in Kleene Normal Form that is uniformly equivalent to $\theta_0(\bar{m}, m', X)$ over $\mathrm{PA}^- + \mathrm{Coll}(\Sigma^0_1{\restriction}\mathrm{KNF})$ as given by the induction hypothesis, then $\eta(\bar{m}, X{\restriction}\ell)$ can be set to

$$\exists m'{<}\mathrm{len}(X{\restriction}\ell)\ \eta_0(\bar{m}, m', X{\restriction}\ell). \qquad \square$$

Corollary 15.6. $\mathrm{Coll}(\Sigma^0_1)$ and $\mathrm{Coll}(\Sigma^0_1{\restriction}\mathrm{KNF})$ are equivalent over PA^-. $\square$

In the next section, we are going to construct a branch required by Proposition 15.5 as a union of (coded sets of) coded binary sequences. The Kleene Normal Form Theorem will be important in this construction because it tells us we can completely determine the Σ^0_1-properties of our branch using its coded initial segments.

15.3 Forcing with nonempty Π^0_1 classes

This whole section is devoted to a proof of Proposition 15.5. We employ a method that resembles a kind of recursion-theoretic forcing devised by Jockusch and Soare [6] which is usually referred to as *forcing with nonempty* Π^0_1 *classes*. Roughly speaking, a Π^0_1 *class* is the set of unbounded branches in some recursive tree.

Fix a countable $(M, \mathscr{X}) \models \mathrm{I}\Sigma^0_1$ and an unbounded binary tree $T \in \mathscr{X}$. To show Proposition 15.5, we build by recursion unbounded binary trees

$$T = T_0 \supseteq T_1 \supseteq T_2 \supseteq \cdots$$

in $\mathscr{X}$ such that $B = \bigcap_{i\in\mathbb{N}} T_i$ is an unbounded branch in T and $(M, \mathscr{X} \cup \{B\}) \models \mathrm{I}\Sigma^0_1$. This can be achieved with the help of two claims, the first of which ensures B is an unbounded branch, and the second of which ensures $(M, \mathscr{X} \cup \{B\}) \models \mathrm{I}\Sigma^0_1$. In view of Lemma 15.4, we may assume $(M, \mathscr{X}) \models \mathrm{RCA}_0$ without loss of generality. Define

- $\mathscr{T} = \{U \in \mathscr{X} : U \text{ is an unbounded binary tree}\}$; and
- $U[\ell] = \{\sigma \in U : \mathrm{len}\,\sigma = \ell\}$ for all $U \in \mathscr{T}$.

Claim 15.5.1. $\forall \ell{\in}M\ \forall U{\in}\mathscr{T}\ \exists V{\in}\mathscr{T}\ (V \subseteq U \text{ and } \exists! \sigma{\in}V\ \mathrm{len}\,\sigma = \ell)$.

Proof of claim. Since U is unbounded, it has an element of length b for every $b \in M$. By restricting such an element to ℓ, we see that

$$(M, \mathscr{X}) \models \forall b\ \exists \sigma{\in}U[\ell]\ \forall \ell'{\in}[\ell, b]\ \underbrace{\exists \tau{\in}U[\ell']\ \tau \supseteq_\mathrm{p} \sigma}_{\Pi^0_1}.$$

Lemma 3.7(c) implies the quantifier for σ in the formula above can be bounded by $2^{\ell+1}$. Therefore, since $(M, \mathscr{X}) \models \mathrm{B}\Sigma^0_1$,

$$(M, \mathscr{X}) \models \exists \sigma{\in}U[\ell]\ \forall \ell'{\geqslant}\ell\ \exists \tau{\in}U[\ell']\ \tau \supseteq_\mathrm{p} \sigma.$$

If $\sigma \in U$ witnessing this, then $V = \{\tau \in U : M \models \tau \subseteq_\mathrm{p} \sigma \vee \sigma \subseteq_\mathrm{p} \tau\}$ is what we want, observing that $V \in \mathscr{X}$ by Δ^0_1-CA. $\dashv$

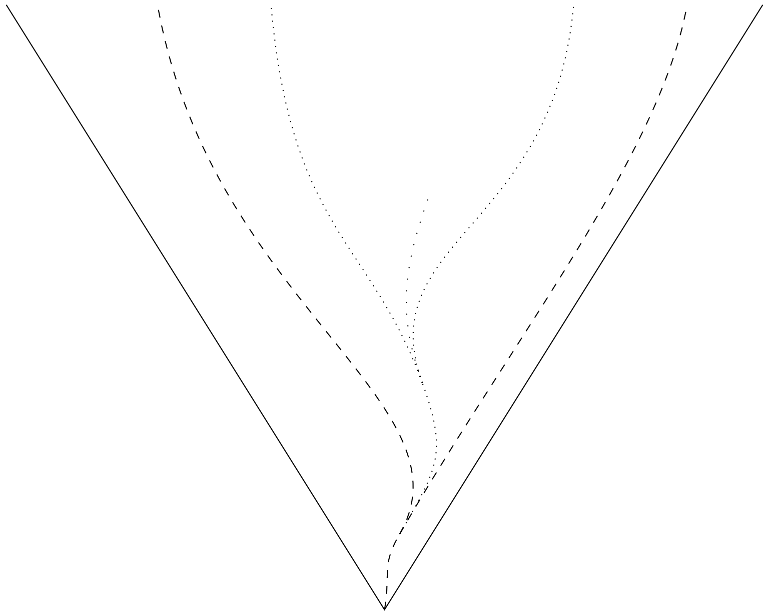

Figure 15.2: Forcing with nonempty Π^0_1 classes

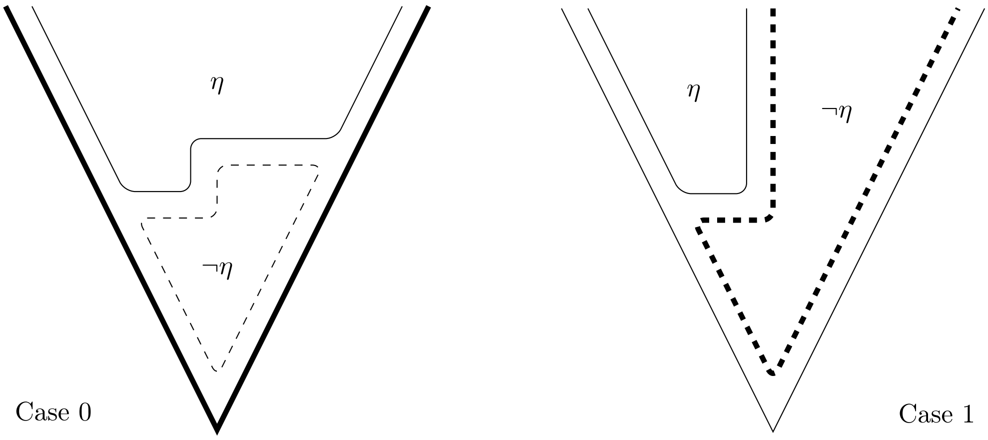

Figure 15.3: Deciding a Σ^0_1-formula $\exists\ell\ \eta(X{\restriction}\ell)$ in Kleene Normal Form

Suppose we are given $U \in \mathscr{T}$. Consider a $\Sigma^0_1(M, \mathscr{X})$-formula $\exists\ell\ \eta(X{\restriction}\ell)$ in Kleene Normal Form. Define $V \subseteq U$ as follows.

Case 0. If $\{\sigma \in U : (M, \mathscr{X}) \models \neg\eta(\sigma)\} \not\subseteq_{\mathrm{cf}} U$, then set $V = U$.

Case 1. If $\{\sigma \in U : (M, \mathscr{X}) \models \neg\eta(\sigma)\} \subseteq_{\mathrm{cf}} U$, then set $V = \{\sigma \in U : (M, \mathscr{X}) \models \neg\eta(\sigma)\}$.

Notice $\{\sigma \in U : (M, \mathscr{X}) \models \neg\eta(\sigma)\}$ is closed under taking initial segments because of the monotonicity property $(*)$ of η. So $V \in \mathscr{T}$ in either case. There are two possible scenarios. Here it is convenient to identify an unbounded branch $B \subseteq V$ with $\{v \in M : \exists\sigma{\in}B\ \sigma(v) = 1\}$, where $\sigma(v) = 1$ stands for $v < \operatorname{len}\sigma \wedge v \in \operatorname{Ack}(\sigma)$. This identification makes sense because the two are mutually Δ^0_1-definable.

Case 0, continued. Now $(M, \mathscr{X}) \models \eta(\sigma)$ for all long enough $\sigma \in U$. So every unbounded branch $B \subseteq V$ makes $(M, \mathscr{X} \cup \{B\}) \models \exists\ell\ \eta(B{\restriction}\ell)$ because $B{\restriction}\ell \in B \subseteq V = U$.

Case 1, continued. Every unbounded branch $B \subseteq V$ makes $(M, \mathscr{X} \cup \{B\}) \models \forall\ell\ \neg\eta(B{\restriction}\ell)$ because $B{\restriction}\ell \in B \subseteq V$.

So, in a sense, the truth of $\exists\ell\ \eta(X\lceil\ell)$ is already *decided* within V. The argument above shows that every tree in $\mathscr{T}$ can be refined to one that decides a given $\Sigma^0_1(M,\mathscr{X})$-formula in Kleene Normal Form. With $\mathrm{I}\Sigma^0_1$, we can actually decide codedly many such formulas in one go.

Claim 15.5.2. Let $a \in M$ and $\theta(m, X) \in \Sigma^0_1(M, \mathscr{X})$. Then for every $U \in \mathscr{T}$, there exists $V \in \mathscr{T}$ which is a subtree of U such that whenever B is an unbounded branch in V,

$$\{m < a : (M, \mathscr{X} \cup \{B\}) \models \theta(m, B)\} \in \mathrm{Cod}(M).$$

Proof of claim. It suffices to show the claim for those $\theta(m, X)$ in Kleene Normal Form $\exists\ell\ \eta(m, X\lceil\ell)$, because the proof of Theorem 2.2 can easily be modified to show $\mathrm{Coll}(\Sigma^0_1\lceil\mathrm{KNF})$ from $\mathrm{I}\Delta_0$ plus $\Sigma^0_1\lceil$KNF-separation in the sense of Theorem 2.7. The plan is to decide $\theta(m, X)$ for every $m < a$ in one go as anticipated in the comment before the statement of the claim, so that the ground model $(M, \mathscr{X})$ already sees the set to be coded. We achieve this by iterating the decision method described above a-many times internally in $(M, \mathscr{X})$. More precisely, let Seq_2 denote the set of all codes for binary sequences, and define

$$W = \big\{\langle\rho,\sigma\rangle \in \mathrm{Seq}_2[a] \times U : (M,\mathscr{X}) \models \underbrace{\forall m{<}a\ \big(\rho(m) = 1 \to \neg\eta(m,\sigma)\big)}_{\Delta^0_0}\big\}.$$

Then $W \in \mathscr{X}$ by Δ^0_1-CA. Notice $(W)_\rho = \{\sigma \in U : \langle\rho,\sigma\rangle \in W\}$ is a binary tree for every $\rho \in \mathrm{Seq}_2[a]$ because η satisfies property $(*)$. These $(W)_\rho$'s are the 2^a-many cases in defining V. To find out which case we are in, we use $\mathrm{I}\Sigma^0_1$ as follows. Observe that

$$\begin{aligned} S &= \{\rho \in \mathrm{Seq}_2[a] : (W)_\rho \text{ is unbounded}\}\\ &= \{\rho \in \mathrm{Seq}_2[a] : (M,\mathscr{X}) \models \underbrace{\forall\ell\ \exists\sigma{\in}\mathrm{Seq}_2[\ell]\ \langle\rho,\sigma\rangle \in W}_{\Pi^0_1}\} \in \mathrm{Cod}(M)\end{aligned}$$

by (a second-order version of) Theorem 2.7. We know $S \neq \varnothing$ because it contains the code for the constant-0 sequence $a \to 2$. Let ρ be the (lexicographically) maximum element of S, which exists by Lemma 2.6(b), and let $V = (W)_\rho$. Then $V \in \mathscr{T}$ by the definition of S. Take any unbounded branch $B \subseteq V$. We show

$$\{m < a : (M, \mathscr{X} \cup \{B\}) \models \theta(m, B)\} = \{m < a : (M,\mathscr{X}) \models \underbrace{\exists\ell\ \forall\sigma{\in}V[\ell]\ \eta(m,\sigma)}_{\Sigma^0_1}\},$$

which is sufficient for the claim, because $\mathrm{I}\Sigma^0_1$ in $(M, \mathscr{X})$ ensures the right-hand side is in $\mathrm{Cod}(M)$ by Theorem 2.7. Pick $m < a$. If $(M,\mathscr{X}) \models \exists\ell\ \forall\sigma{\in}V[\ell]\ \eta(m,\sigma)$, then $(M, \mathscr{X} \cup \{B\}) \models \exists\ell\ \eta(m, B\lceil\ell)$ because $B\lceil\ell \in B \subseteq V$ and $\mathrm{len}(B\lceil\ell) = \ell$. Conversely, suppose $(M,\mathscr{X}) \models \forall\ell\ \exists\sigma{\in}V[\ell]\ \neg\eta(m,\sigma)$. Then $\{\sigma \in V : (M,\mathscr{X}) \models \neg\eta(m,\sigma)\}$ is an unbounded subtree of V, and so

$$\begin{aligned} &\rho(m) = 1 && \text{by the maximality of } \rho,\\ \therefore\quad &(M,\mathscr{X}) \models \forall\sigma{\in}V\ \neg\eta(m,\sigma) && \text{by the definition of } W,\\ \therefore\quad &(M,\mathscr{X}) \models \forall\sigma{\in}B\ \neg\eta(m,\sigma) && \text{since } B \subseteq V,\\ \therefore\quad &(M,\mathscr{X}) \models \forall\ell\ \neg\eta(m,B\lceil\ell) && \text{since } B\lceil\ell \in B,\end{aligned}$$

as required. ⊣

Having proved these two claims, we proceed as follows. Suppose $T_i \in \mathscr{T}$ is found, where i is even. Consider $\ell \in M$, which comes from some fixed enumeration of M of length ω. Apply Claim 15.5.1 to find $T_{i+1} \subseteq T_i$ in $\mathscr{T}$ in which there is a unique node σ of length ℓ. Then $\sigma \in B$ because any binary tree $T \subseteq T_{i+1} \setminus \{\sigma\}$ is bounded. By the uniqueness of σ, no other node of length ℓ can be in B. Repeating this for every $\ell \in M$ thus ensures B contains a unique node at every level $\ell \in M$. Hence B is a branch.

Take $a \in M$ and $\theta(m, X) \in \Sigma^0_1(M, \mathscr{X})$. Since $(M, \mathscr{X})$ is countable, there are only countably many such pairs (a, θ). So we can deal with each and every of them during the construction. Using Claim 15.5.2, find $T_{i+2} \subseteq T_{i+1}$ in $\mathscr{T}$ such that

$$\{m < a : (M, \mathscr{X} \cup \{B\}) \models \theta(m, B)\} \in \mathrm{Cod}(M).$$

Then thanks to Theorem 2.7, the extension $(M, \mathscr{X} \cup \{B\}) \models \mathrm{I}\Sigma^0_1$ at the end. This completes the proof of Proposition 15.5. □

Further exercises

Let us repeat the arguments in this lecture for theories at the level of $\mathrm{B}\Sigma_1 + \exp$. First, we establish an analogue of the Kleene Normal Form Theorem for Δ^0_0-formulas.

Proposition 15.7. For every $\theta(\bar{m}, X) \in \Delta^0_0$, there exists $\eta(\bar{m}, X{\restriction}\ell) \in \Delta^0_0$ satisfying condition $(*)$ such that

$$\mathrm{PA}^- \vdash \forall X\ \forall a\ \forall^\infty b\ \forall \bar{m}{<}a\ \big(\theta(\bar{m}, X) \leftrightarrow \eta(\bar{m}, X{\restriction}b)\big).$$

Here θ may contain undisplayed free variables, in which case η can also contain them.

(a) By imitating the proof of the Kleene Normal Form Theorem, or otherwise, show Proposition 15.7. Notice for every $\mathscr{L}_\mathrm{A}$-term t,

$$\mathrm{PA}^- \vdash \forall \bar{m}{<}a\ \big(t(\bar{m}) \leqslant t(a, a, \ldots, a)\big).$$

The following is the analogue of Proposition 15.5 for $\mathrm{B}\Sigma^0_1 + \exp$.

Theorem 15.8 (Simpson–Smith [12])**.** Take any countable $(M, \mathscr{X}) \models \mathrm{B}\Sigma^0_1 + \exp$. Let T be an unbounded binary tree in $\mathscr{X}$. Then there is an unbounded branch $B \subseteq T$ such that $(M, \mathscr{X} \cup \{B\}) \models \mathrm{B}\Sigma^0_1 + \exp$.

Proof. We proceed as in the proof of Proposition 15.5, in which trees $T = T_0 \supseteq T_1 \supseteq T_2 \supseteq \cdots$ are constructed by recursion.

(b) Explain why we may assume $(M, \mathscr{X}) \models \Delta^0_1\text{-CA}$ without loss of generality.

Using Claim 15.5.1, we can ensure $B = \bigcap_{i \in \mathbb{N}} T_i$ is an unbounded branch in T.

(c) Apply Proposition 15.7 to show that $(M, \mathscr{X} \cup \{B\}) \models \mathrm{I}\Delta^0_0$.

In view of the Kleene Normal Form Theorem, it remains to show how to ensure $(M, \mathscr{X} \cup \{B\}) \models \mathrm{Coll}(\Sigma^0_1{\restriction}\mathrm{KNF})$. This is achieved via the following claim, in which the meaning of $\mathscr{T}$ is the same as that in the proof of Proposition 15.5. Fix $a \in M$ and $\exists \ell\ \eta(m, n, X{\restriction}\ell) \in \Sigma^0_1{\restriction}\mathrm{KNF}(M, \mathscr{X})$.

Claim 15.8.1. For every $U \in \mathscr{T}$, there exists a subtree $V \subseteq U$ which is in $\mathscr{D}_0$ or $\mathscr{D}_1$, where

$$\mathscr{D}_0 = \{U \in \mathscr{T} : \exists m{<}a\ \forall \sigma{\in}U\ \forall n{\leqslant}\mathrm{len}\,\sigma\ \neg\eta(m, n, \sigma)\}, \text{ and}$$
$$\mathscr{D}_1 = \{U \in \mathscr{T} : \text{no } V \subseteq U \text{ is in } \mathscr{D}_0\}.$$

(d) Explain why Claim 15.8.1 is true.

According to Claim 15.8.1, we can carry out the construction such that $T_i \in \mathscr{D}_0 \cup \mathscr{D}_1$ for some $i \in \mathbb{N}$. Fix one such i.

(e) Suppose $T_i \in \mathscr{D}_0$. Show that $(M, \mathscr{X} \cup \{B\}) \models \exists m{<}a\ \forall n\ \forall \ell\ \neg\eta(m, n, B{\restriction}\ell)$.

(f) Suppose $T_i \in \mathscr{D}_1$. Show that $V_m = \{\sigma \in U : (M, \mathscr{X}) \models \forall n{\leqslant}\mathrm{len}\,\sigma\ \neg\eta(m, n, \sigma)\}$ is a bounded binary tree for each $m < a$. Then apply $\mathrm{B}\Sigma^0_1$ in $(M, \mathscr{X})$ to bound these bounds, and conclude that $(M, \mathscr{X} \cup \{B\}) \models \exists b\ \forall m{<}a\ \exists n{<}b\ \exists \ell\ \eta(m, n, B{\restriction}\ell)$ in this case. □

Further comments

Many different proofs of the Π^1_1-conservativity of WKL_0 over RCA_0 are known. One can use, for example, indicators as in Paris [8], special definable sets as in Hájek [5] or in Belanger [2], self-embeddings as in Kaye [unpublished] or in Yokoyama [15], or the Arithmetized Completeness Theorem as in Wong [14].

Combining Lemma 15.4 and Theorem 15.3, one sees that every countable model of $\mathrm{I}\Sigma_1$ is the first-order part of some model of WKL_0. This is actually true for uncountable models too [5, 2]. Consequently, Corollary 4.11 tells us that every model of $\mathrm{I}\Sigma_1$ has a proper end extension satisfying $\mathrm{I}\Delta_0 + \exp$. Notice for countable (nonstandard) models, we can actually require the extension to satisfy $\mathrm{I}\Sigma_1$; see Remark 8.7.

All these results relativize to higher levels of the arithmetical hierarchy; see Avigad [1], Belanger [2], Hájek [5], and Paris [8]. Extensions of $\mathscr{L}_{\mathrm{II}}$-structures that do not add new first-order objects are sometimes called *ω-extensions*. They have proved to be useful in the reverse mathematical study of combinatorial principles related to Ramsey's Theorem for pairs [4]. As observed by Avigad [1], one can cook up $\mathscr{L}_{\mathrm{II}}$-theories T_1, T_2 such that T_2 is a Π^1_1-conservative extension of T_1, but some model of T_1 has no ω-extension satisfying T_2.

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