MODEL THEORY OF ARITHMETIC

Lecture 2: Collection

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> What has this to do with models of arithmetic? More than meets the eye.
>
> Wilfrid Hodges [5, page 181]

The main aim of this lecture is to show that collection is fact a form of induction.

2.1 Strong fragments of arithmetic

Let us put together all the algebraic facts about $\mathbb{N}$ into our base theory.

Definition. The theory of *non-negative parts of discretely ordered commutative rings* (PA^-) is axiomatized by

(i) $\forall x, y, z\ \big((x+y)+z = x+(y+z)\big)$;

(ii) $\forall x, y\ (x+y = y+x)$;

(iii) $\forall x, y, z\ \big((x\times y)\times z = x\times(y\times z)\big)$;

(iv) $\forall x, y\ (x\times y = y\times x)$;

(v) $\forall x, y, z\ (x\times(y+z) = x\times y + x\times z)$;

(vi) $\forall x\ (x+0 = x)$;

(vii) $\forall x\ (x\times 0 = 0 \wedge x\times 1 = x)$;

(viii) $\forall x, y, z\ (x<y \wedge y<z \rightarrow x<z)$;

(ix) $\forall x\ (x \not< x)$;

(x) $\forall x, y\ (x<y \vee x=y \vee y<x)$;

(xi) $\forall x, y, z\ (x<y \rightarrow x+z<y+z)$;

(xii) $\forall x, y, z\ (z>0 \wedge x<y \rightarrow x\times z < y\times z)$;

(xiii) $\forall x, y\ \big(x<y \rightarrow \exists z\ (y = x+z+1)\big)$;

(xiv) $0<1 \wedge \forall x\ (x>0 \rightarrow x \geqslant 1)$;

(xv) $\forall x\ (x \geqslant 0)$.

Induction distinguishes arithmetic from algebra. It is the defining property of $\mathbb{N}$.

Definition. Let Γ be a class of $\mathscr{L}_{\mathrm{A}}$-formulas. The theory $\mathrm{I}\Gamma$ consists of the axioms of PA^- and

$$\forall \bar{z}\ \Big(\theta(0,\bar{z}) \wedge \forall x\ \big(\theta(x,\bar{z}) \rightarrow \theta(x+1,\bar{z})\big) \rightarrow \forall x\ \theta(x,\bar{z})\Big)$$

for all $\theta \in \Gamma$. *Peano arithmetic* (PA) is $\bigcup_{n\in\mathbb{N}} \mathrm{I}\Sigma_n$.

The induction scheme can be paraphrased as: if a definable set contains 0 and is closed under successor, then it must contain all numbers.

There are various ways in which pairs of natural numbers (x, y) can be coded into a single $z = \langle x, y\rangle$. For concreteness, let us fix the following.

Definition (Cantor [3])**.** $\langle x, y\rangle = \frac{1}{2}(x + y)(x + y + 1) + y$.

For the purposes of this course, it does not matter exactly how pairing is defined as long as $\langle x, y\rangle = z$ is Δ_0, and the following hold.

Lemma 2.1. $\mathrm{I}\Delta_0$ proves

(a) $\forall x, y, z\ \big(z = \langle x, y\rangle \rightarrow x \leqslant z \wedge y \leqslant z\big)$;

(b) $\forall x, y\ \exists! z\ (z = \langle x, y\rangle)$; and

(c) $\forall z\ \exists! x, y\ \big(z = \langle x, y\rangle\big)$.

Proof. Exercise. The bound in part (a) helps make $\mathrm{I}\Delta_0$ applicable. □

Pairing extends iteratively to the coding of $(k + 2)$-tuples for every $k \in \mathbb{N}$ by setting

$$\langle x_1, x_2, \ldots, x_{k+1}, x_{k+2}\rangle = \langle\langle x_1, x_2, \ldots, x_{k+1}\rangle, x_{k+2}\rangle.$$

This allows one to 'contract' unbounded quantifiers of the same kind *without affecting quantifier complexity*, e.g., if $n \in \mathbb{N}$ and $\theta \in \Pi_n$, then we can rewrite the Σ_{n+1}-formula $\exists \bar{x}\ \theta(\bar{x}, \bar{z})$ as

$$\exists w\ \forall \bar{x}{\leqslant}w\ \big(w = \langle \bar{x}\rangle \wedge \theta(\bar{x}, \bar{z})\big),$$

and the formula would stay Σ_{n+1}. As a result, blocks of unbounded quantifiers can always be assumed to be of length one whenever our model satisfies $\mathrm{I}\Delta_0$.

Our first application of this is to show induction implies collection.

Definition. $\mathrm{B}\Sigma_n = \mathrm{I}\Delta_0 + \mathrm{Coll}(\Sigma_n)$ for all $n \in \mathbb{N}$.

The letter B stands for *bounding* here. Be aware that $\mathrm{PA}^- + \bigcup_{n\in\mathbb{N}} \mathrm{Coll}(\Sigma_n) \nvdash \mathrm{I}\Delta_0$: see Exercise 7.7 in Kaye's book [7].

Definition. Let M be an $\mathscr{L}_\mathrm{A}$-structure. Denote by $\mathscr{L}_\mathrm{A}(M)$ the language obtained from $\mathscr{L}_\mathrm{A}$ by adding a new constant symbol for every element of M. The structure M expands naturally to an $\mathscr{L}_\mathrm{A}(M)$-structure. The classes $\Sigma_n(M), \Pi_n(M), \Delta_n(M)$ are defined as in the usual arithmetic hierarchy, except that we now allow the new constant symbols to appear in the formulas.

Theorem 2.2 (Parsons [11])**.** $\mathrm{I}\Sigma_{n+1} \vdash \mathrm{B}\Sigma_{n+1}$ for every $n \in \mathbb{N}$.

Proof. We proceed by (an external) induction on $n \in \mathbb{N}$. Suppose $\mathrm{I}\Sigma_{m+1} \vdash \mathrm{B}\Sigma_{m+1}$ for all $m < n \in \mathbb{N}$. Let $M \models \mathrm{I}\Sigma_{n+1}$. Take any $a \in M$ and any $\varphi(v, x, y) \in \Pi_n(M)$ such that

$$M \models \forall x{<}a\ \exists y\ \exists v\ \varphi(v, x, y). \qquad (*)$$

Thanks to the help of pairing, the blocks of quantifiers can be assumed to consist only of single variables. We claim that

$$M \models \forall t{\leqslant}a\ \underbrace{\exists b\ \underbrace{\forall x{<}t\ \exists y, v{<}b\ \underbrace{\varphi(v, x, y)}_{\Pi_n}}_{\Pi_n \text{ over } \mathrm{B}\Sigma_n}}_{\Sigma_{n+1} \text{ over } \mathrm{B}\Sigma_n},$$

which suffices to finish the proof since $a \leqslant a$. If $n = 0$, then $\varphi \in \Delta_0$ and so $\mathrm{B}\Sigma_n$ is actually not needed in the complexity calculation above. If $n = m + 1$, then $M \models \mathrm{I}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n = \mathrm{I}\Sigma_{m+1} \vdash \mathrm{B}\Sigma_{m+1} = \mathrm{B}\Sigma_n$ by the induction hypothesis. So in any case, the formula inside the largest curly bracket above is Σ_{n+1} over M. We can thus use $\mathrm{I}\Sigma_{n+1}$ internally in M to prove our claim.

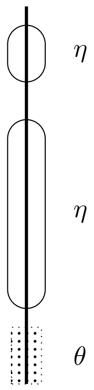

Figure 2.1: Showing $\mathrm{I}\Sigma_n \vdash \mathrm{L}\Pi_n$

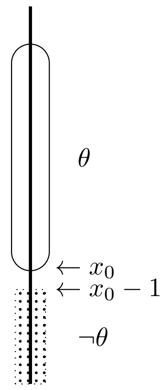

Figure 2.2: Showing $\mathrm{L}\Sigma_n \vdash \mathrm{I}\Pi_n$

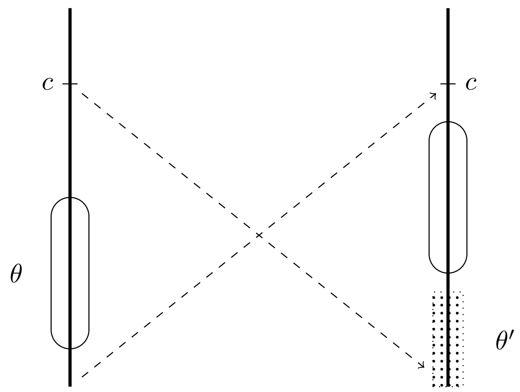

Figure 2.3: Showing $\mathrm{I}\Sigma_n \vdash \mathrm{I}\Pi_n$

The case when $t = 0$ is trivial because no $x \in M$ is less than 0. Suppose $t < a$ and $b \in M$ such that $M \models \forall x{<}t\ \exists y, v{<}b\ \varphi(v, x, y)$. Using our hypothesis $(*)$, find $y_0, v_0 \in M \models \varphi(v_0, t, y_0)$. Setting $b' = \max\{b, v_0, y_0\} + 1$ gives

$$M \models \forall x{<}t+1\ \exists y, v{<}b'\ \varphi(v, x, y),$$

completing the induction. □

This proof is a nice demonstration of the beautiful interplay between internal and external induction. This seems to be a characteristic of nonstandard arithmetic, and cannot even be found in mainstream set theory. Notice we would run into problems in this proof if we allow the variable x in line $(*)$ to be a tuple of length more than one. It is known [10] that $\mathrm{I}\Sigma_0 \nvdash \mathrm{B}\Sigma_0$.

Instead of proving induction from collection straightaway, we take a detour.

Definition. Let Γ be a set of $\mathscr{L}_\mathrm{A}$-formulas. Then $\mathrm{L}\Gamma$ consists of the axioms of PA^- and

$$\forall \bar{z}\ \big(\exists x\ \eta(x, \bar{z}) \to \exists x\ \big(\eta(x, \bar{z}) \land \forall x'{<}x\ \neg\eta(x', \bar{z})\big)\big)$$

for all $\eta \in \Gamma$.

The $\mathrm{L}\Gamma$s refer to the *least number principle*, which says that every nonempty set of natural numbers has a least element. It is well-known that this principle is the contrapositive of induction, but the level-by-level equivalence appears to be non-trivial, especially if we want to avoid mentioning collection.

Theorem 2.3 (Paris–Kirby [10])**.** $\mathrm{I}\Sigma_n$, $\mathrm{I}\Pi_n$, $\mathrm{L}\Sigma_n$ and $\mathrm{L}\Pi_n$ are equivalent for every $n \in \mathbb{N}$.

Proof. ($\mathrm{I}\Sigma_n \vdash \mathrm{L}\Pi_n$.) Let $M \models \mathrm{I}\Sigma_n$. Take $\eta(x) \in \Pi_n(M)$ such that

$$M \models \forall x \ \big(\eta(x) \to \exists x'{<}x \ \eta(x')\big). \tag{†}$$

Define $\theta(x)$ to be $\forall x'{\leqslant}x \ \neg\eta(x')$. It is Σ_n over M because either $n = 0$ or $M \models \mathrm{B}\Sigma_n$ by Theorem 2.2. So $\mathrm{I}\Sigma_n$ applies to θ. Notice $M \models \theta(0)$, since otherwise $M \models \eta(0)$, which is not possible by (†). Suppose $x \in M \models \neg\theta(x+1)$. First find $x' \leqslant x+1$ such that $M \models \eta(x')$ by unravelling θ. Then, using assumption (†), find $x'' < x'$ such that $M \models \eta(x'')$. Now $x'' < x' \leqslant x+1$, implying $x'' \leqslant x$. So $M \models \neg\theta(x)$. By $\mathrm{I}\Sigma_n$, we conclude $M \models \forall x \ \theta(x)$. This makes $M \models \neg\exists x \ \eta(x)$.

($\mathrm{I}\Pi_n \vdash \mathrm{L}\Sigma_n$.) Proceed as in the previous paragraph. We do not know $\mathrm{I}\Pi_n \vdash \mathrm{B}\Sigma_n$ for $n > 0$ yet, but this is not needed because the bounded universal quantifier in θ is now added in front of an unbounded quantifier of the same kind.

($\mathrm{L}\Sigma_n \vdash \mathrm{I}\Pi_n$.) Let $M \models \mathrm{L}\Sigma_n$. Pick any $\theta(x) \in \Pi_n(M)$ such that $M \models \exists x \ \neg\theta(x)$. Using $\mathrm{L}\Sigma_n$, find $x_0 \in M \models \neg\theta(x_0) \wedge \forall x{<}x_0 \ \theta(x)$. If $x_0 = 0$, then $M \models \neg\theta(0)$. If $x_0 > 0$, then $x_0 - 1$ exists, and $M \models \theta(x_0 - 1)$ by minimality. In either case, we are done.

($\mathrm{L}\Pi_n \vdash \mathrm{I}\Sigma_n$.) Follow the argument in the previous paragraph.

($\mathrm{I}\Sigma_n \vdash \mathrm{I}\Pi_n$.) Let $M \models \mathrm{I}\Sigma_n$. Take $\theta(x) \in \Pi_n(M)$ such that $M \models \exists x \ \neg\theta(x)$. We want

$$\text{either} \quad M \models \neg\theta(0) \tag{1}$$
$$\text{or} \quad M \models \exists x \ \big(\theta(x) \wedge \neg\theta(x+1)\big). \tag{2}$$

Pick $c \in M \models \neg\theta(c)$. Define $\theta'(x)$ to be $\neg\theta(c-x)$, i.e.,

$$x \leqslant c \to \underbrace{\exists w{\leqslant}c \ (c = x + w \wedge \underbrace{\neg\theta(w)}_{\Sigma_n})}_{\Sigma_n}.$$

So $\mathrm{I}\Sigma_n$ is applicable to θ'. Notice $M \models \theta'(0)$ because $M \models \neg\theta(c)$. If $M \models \forall x \ (\theta'(x) \to \theta'(x+1))$, then $M \models \forall x \ \theta'(x)$, which implies $M \models \theta'(c)$ and thus $M \models \neg\theta(0)$, giving condition (1). So suppose not. Let $x \in M \models \theta'(x) \wedge \neg\theta'(x+1)$. The second conjunct implies $x + 1 \leqslant c$, so that $x \leqslant c$. Substituting these values back into θ' makes $M \models \neg\theta(c-x) \wedge \theta(c-(x+1))$. Therefore, condition (2) is true.

($\mathrm{I}\Pi_n \vdash \mathrm{I}\Sigma_n$.) Use the argument in the previous paragraph. The formula θ' remains Π_n because it can be written as

$$x \leqslant c \to \forall w{\leqslant}c \ (c = x + w \to \neg\theta(w)).$$ □

2.2 The Ackermann interpretation

The reason why we do not need more function symbols in our language $\mathscr{L}_{\mathrm{A}}$, say one for exponentiation, is because we can already define most operations of interest using only $+$ and $\times$. For example, with some coding, one can express $y = 2^x$ as a Σ_1-formula. With more dirty work, one can actually make this Δ_0.

Fact 2.4 (Bennett [2])**.** There is a Δ_0-formula $\varepsilon(x, y)$ such that $\mathrm{I}\Delta_0$ proves

(a) $\forall x, y, y' \ \big(\varepsilon(x, y) \wedge \varepsilon(x, y') \to y = y'\big)$;

(b) $\varepsilon(0, 1) \wedge \forall x, y \ \big(\varepsilon(x, y) \leftrightarrow \varepsilon(x+1, 2y)\big)$.

Proof. Outside the scope of this course. See Gaifman–Dimitracopoulos [4]. □

It is known [9] that $\mathrm{I}\Delta_0$ can only prove the totality of functions of polynomial growth. So $\mathrm{I}\Delta_0$ does not prove the totality of exponentiation. Nevertheless, we can state it as an extra assumption if needed. The totality of exponentiation can easily be proved using Σ_1 induction.

Definition. Let exp be the sentence $\forall x \ \exists y \ \varepsilon(x, y)$, where ε is a formula provided by Fact 2.4. Write $y = 2^x$ for $\varepsilon(x, y)$ from now on.

Exponentiation is essential for the (carefree) coding of sets. For example, if $a \in M \models \mathrm{I}\Delta_0$ and if we were to code all subsets of $\{0, 1, \ldots, a\}$ in M, then there must be a total of 2^{a+1} codes.

Our method of coding sets originated from Ackermann [1].

Definition. Let $i \in \mathrm{Ack}(x)$ denote a Δ_0-formula that expresses 'the ith digit in the binary expansion of x is 1', say

$$\exists w{<}x\ \exists p{\leqslant}x\ \exists r{<}p\ \big(p = 2^i \wedge x = (2w+1)p + r\big).$$

Let $M \models \mathrm{I}\Delta_0$. If $c \in M$, then $\mathrm{Ack}(c) = \{i \in M : M \models i \in \mathrm{Ack}(c)\}$. These sets are said to be *coded* in M. The set of all coded subsets of M is denoted $\mathrm{Cod}(M)$.

Example 2.5. Let $a \in M \models \mathrm{I}\Delta_0 + \exp$. Then $a = \{0, 1, \ldots, a-1\}$ is coded by $\underbrace{11\cdots1}_{a\text{-many 1s}}{}_2 = 2^a - 1$.

Coded sets are thought of as nice subsets of the model. Notice all of them are Δ_0-definable.

Lemma 2.6. Let $c \in M \models \mathrm{I}\Delta_0$.

(a) If $i \in \mathrm{Ack}(c)$, then $i < 2^i \leqslant c$.

(b) If $\mathrm{Ack}(c) \neq \varnothing$, then it has a minimum and a maximum.

Proof sketch. For the maximum in part (b), consider the least m such that $[m, c] \cap \mathrm{Ack}(c) = \varnothing$. □

It would be good if many sets are nice, for example, if all definable sets are coded. However, this is not possible because every coded set is bounded above. Fortunately, given enough induction, we have no other obstacle.

Definition. Let M be an $\mathscr{L}_\mathrm{A}$-structure and Γ be a class of $\mathscr{L}_\mathrm{A}$-formulas. A set S is Γ-*definable (with parameters)* in M if

$$S = \{\bar{x} \in M : M \models \theta(\bar{x}, \bar{c})\}$$

for some $\theta \in \Gamma$ and some $\bar{c} \in M$. The collection of all Γ-definable sets in M is denoted $\Gamma\text{-}\mathrm{Def}(M)$. Set $\Delta_n\text{-}\mathrm{Def}(M) = \Sigma_n\text{-}\mathrm{Def}(M) \cap \Pi_n\text{-}\mathrm{Def}(M)$ for every $n \in \mathbb{N}$. If $a \in M$ and $S \subseteq M$, then $S{\upharpoonright}a = \{x \in S : x < a\}$.

Theorem 2.7 (Harvey Friedman)**.** For all $n \in \mathbb{N}$ and all $M \models \mathrm{I}\Delta_0 + \exp$, the following are equivalent.

(a) $M \models \mathrm{I}\Sigma_n$.

(b) $S{\upharpoonright}a \in \mathrm{Cod}(M)$ for every $S \in \Sigma_n\text{-}\mathrm{Def}(M)$ and every $a \in M$.

The analogous statement about Δ_{n+1}-definable sets involves the collection schemes.

Theorem 2.8 (folklore)**.** For all $n \in \mathbb{N}$ and all $M \models \mathrm{I}\Delta_0 + \exp$, the following are equivalent.

(a) $M \models \mathrm{B}\Sigma_{n+1}$.

(b) $S{\upharpoonright}a \in \mathrm{Cod}(M)$ for every $S \in \Delta_{n+1}\text{-}\mathrm{Def}(M)$ and every $a \in M$.

Theorems 2.7 and 2.8 will be proved in the next lecture.

Corollary 2.9 (Paris–Kirby [10])**.** $\mathrm{B}\Sigma_{n+1} + \exp \vdash \mathrm{I}\Sigma_n$ for every $n \in \mathbb{N}$.

Proof. Because $\Delta_{n+1} \supseteq \Sigma_n$ over any model. □

The equivalence of induction and collection in first-order arithmetic should be seen as a happy but rare coincidence. In second-order arithmetic, collection is much weaker than induction. In set theory, induction is much weaker than collection.

$$\begin{array}{lcl} \mathrm{I}\Sigma_{n+1} & & \\ \Downarrow & & \\ \mathrm{B}\Sigma_{n+1} & & \\ \Downarrow & & \\ \mathrm{I}\Sigma_n & \Leftrightarrow & \mathrm{L}\Pi_n \\ \Updownarrow & & \\ \mathrm{I}\Pi_n & \Leftrightarrow & \mathrm{L}\Sigma_n \end{array}$$

Figure 2.4: Implications between induction and collection schemes

Further exercises

Corollary 2.9 actually does not need exp.

Theorem 2.10 (Paris–Kirby [10]). $\mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$ for every $n \in \mathbb{N}$.

Proof. Proceed by induction on $n \in \mathbb{N}$.

(a) Explain why $\mathrm{B}\Sigma_1 \vdash \mathrm{I}\Sigma_0$.

Suppose $n \in \mathbb{N}$ such that $\mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$. Let $M \models \mathrm{B}\Sigma_{n+2}$ and $\theta(x) \in \Sigma_{n+1}(M)$ such that

$$M \models \theta(0) \wedge \forall x \ \big(\theta(x) \rightarrow \theta(x+1)\big).$$

Take any $a \in M$. Write $\theta(x)$ as $\exists \bar{u}\ \eta(\bar{u}, x)$, where $\eta \in \Pi_n(M)$.

(b) Show that $M \models \exists b\ \forall x{\leqslant}a\ \exists \bar{u}{<}b\ \big(\eta(\bar{u}, x) \vee \forall \bar{v}\ \neg\eta(\bar{v}, x)\big)$.

Fix $b \in M \models \forall x{\leqslant}a\ \exists \bar{u}{<}b\ \big(\eta(\bar{u}, x) \vee \forall \bar{v}\ \neg\eta(\bar{v}, x)\big)$.

(c) Deduce that $M \models \forall x{\leqslant}a\ \big(\exists \bar{u}\ \eta(\bar{u}, x) \leftrightarrow \exists \bar{u}{<}b\ \eta(\bar{v}, x)\big)$.

(d) Apply induction on the formula $x \leqslant a \rightarrow \exists \bar{u}{<}b\ \eta(\bar{u}, x)$ to show that $M \models \theta(a)$. □

Further reading

- Although PA^- seems entirely algebraic, it is strong enough for many arithmetic purposes. For instance, it proves all true Σ_1-sentences [7, Chapter 2]. It admits the Incompleteness Theorems [7, 13], and the coding of sequences [6], etc. It can even interpret $\mathrm{I}\Delta_0$ [13, page 286].

- In Theorem 2.3, we noted the delicateness of the level-by-level equivalence between induction principles and least number principles. This delicateness is further demonstrated by the fact that such equivalence at the Δ_1-level is still open.

  Definition. Let $n \in \mathbb{N}$. The theory $\mathrm{I}\Delta_{n+1}$ consists of the axioms of PA^- and

  $$\forall \bar{z}\ \Big(\forall x\ \big(\varphi(x, \bar{z}) \leftrightarrow \psi(x, \bar{z})\big) \wedge \varphi(0, \bar{z}) \wedge \forall x\ \big(\varphi(x, \bar{z}) \rightarrow \varphi(x+1, \bar{z})\big) \rightarrow \forall x\ \varphi(x, \bar{z})\Big)$$

  for all $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$. Similarly $\mathrm{L}\Delta_{n+1}$ consists of the axioms of PA^- and

  $$\forall \bar{z}\ \Big(\forall x\ \big(\varphi(x, \bar{z}) \leftrightarrow \psi(x, \bar{z})\big) \wedge \exists x\ \varphi(x, \bar{z}) \rightarrow \exists x\ \big(\varphi(x, \bar{z}) \wedge \forall x'{<}x\ \neg\varphi(x', \bar{z})\big)\Big)$$

  for all $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$.

  Our proof of Theorem 2.3 shows $\mathrm{L}\Delta_{n+1} \vdash \mathrm{I}\Delta_{n+1}$ for every $n \in \mathbb{N}$. Jeff Paris asked whether the converse holds. This was partially answered by Slaman.

  Theorem 2.11 (Slaman). $\mathrm{I}\Delta_{n+1} + \exp \vdash \mathrm{L}\Delta_{n+1}$ for every $n \in \mathbb{N}$.

Since $I\Sigma_1$ proves exp but $I\Delta_1$ does not, this provides an answer to all but one levels.

Question 2.12 (Jeff Paris)**.** Does $I\Delta_1 \vdash L\Delta_1$?

See Slaman's paper [12] for more information, including a proof of the following.

Theorem 2.13 (Robin Gandy)**.** $L\Delta_{n+1}$ and $B\Sigma_{n+1}$ are equivalent for every $n \in \mathbb{N}$.

- Observe that when proving the equivalence of $I\Sigma_n$ and $I\Pi_n$ in Theorem 2.3, we introduced a new parameter. The introduction of such a parameter cannot be avoided in general, because the parameter-free induction schemes for Σ_n- are Π_n-formulas are no longer equivalent [8], except when $n = 0$.
- It is known that the converses to Theorem 2.2 and Corollary 2.9 do not hold, i.e., we know $B\Sigma_{n+1} \nvdash I\Sigma_{n+1}$ and $I\Sigma_n \nvdash B\Sigma_{n+1} + \exp$ for every $n \in \mathbb{N}$. See Paris–Kirby [10] or Chapter 10 of Kaye's book [7] for the proofs.

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