Lecture 4: The Arithmetized Completeness Theorem

Tin Lok Wong

29 October, 2014

> No list of open problems concerning models of PA is complete without the venerable Scott set Problem.
>
> Roman Kossak and James Schmerl [4, page 289]

4.1 Second-order arithmetic

Let us go back to Scott's theorem which says that all exponential cuts, when expanded by the coded sets, satisfy the Weak König Lemma.

Theorem 3.5 (essentially Scott [5])**.** Let $M \models \mathrm{I}\Delta_0 + \exp$ and I be a proper exponential cut of M. If T is an unbounded binary tree in $\mathrm{Cod}(M/I)$, then it has an unbounded branch $B \in \mathrm{Cod}(M/I)$.

Suppose I and M are as in the statement of the theorem. Then with the structure inherited from M, it can be shown that I is itself a model of $\mathrm{I}\Delta_0 + \exp$. Recall the definition of binary trees and branches involves the relation $\subseteq_{\mathrm{p}}$. So formally speaking, we need to specify in which structure this relation is evaluated. It will turn out that the evaluations in M and in I agree with each other, because $\sigma \subseteq_{\mathrm{p}} \tau$ is Δ_0 by Lemma 3.7(d); see Proposition 4.12 below. For the purpose of this proof, let us interpret $\subseteq_{\mathrm{p}}$ in M. For other purposes, it would be more informative to interpret $\subseteq_{\mathrm{p}}$ in I.

Proof. Let $t \in M$ such that $T = \mathrm{Ack}(t/I)$. Since T is unbounded,

$$\begin{aligned} I &\subseteq \{\mathrm{len}\,\tau : \tau \in \mathrm{Ack}(t) \wedge \forall\sigma{\subseteq_{\mathrm{p}}}\tau\ \sigma \in \mathrm{Ack}(t)\} \\ &= \big\{\ell \in M : M \models \exists\tau{\in}\mathrm{Ack}(t)\ \big(\ell = \mathrm{len}\,\tau \wedge \forall\sigma{\subseteq_{\mathrm{p}}}\tau\ \sigma \in \mathrm{Ack}(t)\big)\big\}. \end{aligned}$$

Using Δ_0-overspill, find $\tau_0 \in \mathrm{Ack}(t)$ such that $\mathrm{len}\,\tau_0 \in M \setminus I$. Notice

$$\{\tau \in \mathrm{Ack}(t) : \tau \subseteq_{\mathrm{p}} \tau_0\} \in \mathrm{Cod}(M)$$

by Δ_0-separation from Theorem 2.7. Let $b \in M$ code this set, and $B = \mathrm{Ack}(b/I)$. Clearly $B \subseteq T$. Lemma 3.7(h) says B is a branch. It remains to show B is unbounded. If $\tau \in B$, then $\mathrm{len}\,\tau < \tau \in I$, making $\mathrm{len}\,\tau \in I$ since I is a cut. Conversely, suppose $\ell \in I$. Then $\tau_0{\restriction}\ell \subseteq_{\mathrm{p}} \tau_0$. Lemma 3.7 implies $\tau_0{\restriction}\ell < 2^{\ell+1}$. So $\tau_0{\restriction}\ell \in I$ because I is an exponential cut. Therefore $\tau_0{\restriction}\ell \in \mathrm{Ack}(b/I) = B$ with $\mathrm{len}(\tau_0{\restriction}\ell) = \ell$, as required. □

This proof is a nice demonstration of how nonstandard methods work: we show a fact about a smaller structure by passing on to a larger structure, and then restrict back to the smaller one.

Theorem 3.5 can be stated much more neatly using the language of *second-order arithmetic.* Note that our logic will always be first-order; it is only the arithmetic that becomes second-order.

Definition. The *language for second-order arithmetic* is denoted $\mathscr{L}_{\mathrm{II}}$. It has a *number sort* and a *set sort*, which are assumed to partition the universe of any $\mathscr{L}_{\mathrm{II}}$-structure. Elements of the number sort are called *first-order objects*, or simply *numbers*. Elements of the set sort are called *second-order objects*, or *sets*. Lowercase Roman letters $n, m, x, y, \ldots$ are reserved for number variables, and uppercase Roman letter $A, B, X, Y, \ldots$ are reserved for set variables. Within the number sort, we have the symbols of $\mathscr{L}_{\mathrm{A}}$ together with equality. There is also a binary relation $\in$ in $\mathscr{L}_{\mathrm{II}}$ that relates a first-order object to a second-order object. Equality on the set sort is *not* a primitive notion in $\mathscr{L}_{\mathrm{II}}$; it is defined via *extensionality*:

$$\forall X, Y\ \big(\forall x\ (x \in X \leftrightarrow x \in Y) \leftrightarrow X = Y\big).$$

An $\mathscr{L}_{\mathrm{II}}$-formula is Δ^0_0 if all its quantifiers are bounded (number) quantifiers. If $n \in \mathbb{N}$, then

$$\Sigma^0_n = \big\{\exists \bar{x}_1\ \forall \bar{x}_2\ \cdots\ \mathrm{Q}\bar{x}_n\ \xi(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, \bar{z}, \bar{Z}) : \mathrm{Q} \in \{\forall, \exists\} \text{ and } \xi \in \Delta^0_0\big\}, \text{ and}$$
$$\Pi^0_n = \big\{\forall \bar{x}_1\ \exists \bar{x}_2\ \cdots\ \mathrm{Q}'\bar{x}_n\ \zeta(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, \bar{z}, \bar{Z}) : \mathrm{Q}' \in \{\forall, \exists\} \text{ and } \zeta \in \Delta^0_0\big\}.$$

An $\mathscr{L}_{\mathrm{II}}$-formula is Δ^0_n if it is equivalent to both a Σ^0_n- and a Π^0_n-formula (over some theory or some $\mathscr{L}_{\mathrm{II}}$-structure). Formulas in $\mathscr{L}_{\mathrm{II}}$ with no set quantifiers are called *arithmetical* or Δ^1_0. Equivalently, we can define $\Delta^1_0 = \bigcup_{n\in\mathbb{N}} \Sigma^0_n$. We write $\mathscr{L}_{\mathrm{II}}$-structures as pairs $(M, \mathscr{X})$, where M is the universe for the number sort, and $\mathscr{X}$ is the universe for the set sort. In this case, we also call M and $\mathscr{X}$ respectively the *first-* and *second-order parts* of $(M, \mathscr{X})$.

Remark 4.1.

- By restricting to the number sort, we view $\mathscr{L}_{\mathrm{A}}$ as a sublanguage of $\mathscr{L}_{\mathrm{II}}$.
- Checking whether two sets $X, Y \subseteq \mathbb{N}$ are equal involves checking infinitely many equivalences

  $$x \in X \leftrightarrow x \in Y,$$

  where x ranges over $\mathbb{N}$. Therefore, even if we have complete bitwise information about X and Y, this cannot be determined within a finite number of computation steps. This matches with the fact that $X = Y$ is Π^0_1. Allowing the equality of sets to be a primitive in $\mathscr{L}_{\mathrm{II}}$ would destroy our recursion-theoretic understanding of the arithmetical hierarchy, most notably Fact 4.3 below.
- Bounded quantifiers are defined as in the case of $\mathscr{L}_{\mathrm{A}}$. In particular, all bounded quantifiers are number quantifiers.
- The superscript 0 in $\Sigma^0_n, \Pi^0_n, \Delta^0_n$ distinguishes these formula classes from their first-order counterparts $\Sigma_n, \Pi_n, \Delta_n$, which are not allowed to involve second-order objects. Similarly, we reserve the word *arithmetic* for formulas that do not involve second-order objects. In other words, the arithmetic formulas are essentially the $\mathscr{L}_{\mathrm{A}}$-formulas.
- Let $(M, \mathscr{X})$ be any $\mathscr{L}_{\mathrm{II}}$-structure. By extensionality, the relation $\sim$ defined by

  $$X \sim Y \quad \Leftrightarrow \quad (M, \mathscr{X}) \models X = Y$$

  is an equivalence relation on $\mathscr{X}$. In the quotient $\mathscr{L}_{\mathrm{II}}$-structure $(M, \mathscr{X}/{\sim})$, equality on the set sort is interpreted as real equality. Moreover $(M, \mathscr{X}/{\sim}) \equiv (M, \mathscr{X})$. So we may assume without loss of (much) generality that equality on the set sort is always interpreted as real equality, and furthermore $\mathscr{X} \subseteq \mathcal{P}(M)$.

Example 4.2. (1) $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ is called the *standard model of second-order arithmetic.*

(2) If $I \subseteq_{\mathrm{e}} M \models \mathrm{I}\Delta_0$, then $(I, \mathrm{Cod}(M/I))$ is an $\mathscr{L}_{\mathrm{II}}$-structure.

Definition. WKL denotes an $\mathscr{L}_{\mathrm{II}}$-sentence that expresses

$$\forall \text{tree } T\ \exists \text{branch } B \subseteq T\ (\forall \ell\ \exists \tau{\in}T\ \operatorname{len}\tau = \ell \rightarrow \forall \ell\ \exists \tau{\in}B\ \operatorname{len}\tau = \ell).$$

All these give another way of stating Scott's Theorem 3.5.

Theorem 3.5 (reformulated)**.** Let $M \models \mathrm{I}\Delta_0 + \exp$ and I be a proper exponential cut of M. Then $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}$.

As in the case of $\mathscr{L}_{\mathrm{A}}$, arithmetical formulas have recursion-theoretic meanings.

Definition. Let $A \subseteq \mathbb{N}$. A *program with oracle* A is defined recursively as the programs $\mathrm{Prog}\langle\theta\rangle$ in Lecture 1, except we allow another family of basic programs $\mathrm{Prog}\langle t(\bar{x}) \in A\rangle$, where t ranges over number terms, that satisfies

$$\mathrm{Prog}\langle t \in A\rangle(\bar{x}) = \begin{cases} \texttt{true}, & \text{if } t(\bar{x}) \in A; \\ \texttt{false}, & \text{otherwise,} \end{cases}$$

for all $\bar{x} \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then $S \subseteq \mathbb{N}^k$ is *recursively enumerable in* A, or *r.e. in* A, if

$$S = \{\bar{x} \in \mathbb{N}^k : P(\bar{x}) = \texttt{true}\}$$

for some program P with oracle A. The set S is *recursive in* A if both S and $\mathbb{N}^k \setminus S$ are r.e. in A.

Informally speaking, a set S being r.e. or recursive in another set A implies that one can extract information about S using information about A. Therefore, we may view such A as possessing at least as much information as S does.

By a similar proof, one can obtain an analogue of Corollary 1.7 for the arithmetical hierarchy.

Fact 4.3. Let $A \subseteq \mathbb{N}$.

(a) The sets r.e. in A are exactly those that are Σ^0_1-definable in $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ with A as the only set parameter.

(b) The sets recursive in A are exactly those that are Δ^0_1-definable in $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ with A as the only set parameter. □

A main difference in the axiom schemes in $\mathscr{L}_{\mathrm{II}}$ compared to those in $\mathscr{L}_{\mathrm{A}}$ is the appearance of set parameters.

Definition.
- Let Γ be a set of $\mathscr{L}_{\mathrm{II}}$-formulas. Then $\mathrm{I}\Gamma$ consists of all sentences of the form

$$\forall \bar{z}\ \forall \bar{Z}\ \big(\theta(0, \bar{z}, \bar{Z}) \wedge \forall x\ \big(\theta(x, \bar{z}, \bar{Z}) \rightarrow \theta(x+1, \bar{z}, \bar{Z})\big) \rightarrow \forall x\ \theta(x, \bar{z}, \bar{Z})\big),$$

  where $\theta \in \Gamma$.
- The theory Δ^0_1-CA consists of all sentences of the form

$$\forall \bar{z}\ \forall \bar{Z}\ \big(\forall x\ \big(\varphi(x, \bar{z}, \bar{Z}) \leftrightarrow \psi(x, \bar{z}, \bar{Z})\big) \rightarrow \exists X\ \forall x\ \big(x \in X \leftrightarrow \varphi(x, \bar{z}, \bar{Z})\big)\big),$$

  where $\varphi \in \Sigma^0_1$ and $\psi \in \Pi^0_1$.
- $\mathrm{RCA}_0 = \mathrm{PA}^- + \mathrm{I}\Sigma^0_1 + \Delta^0_1\text{-CA}$.
- $\mathrm{WKL}_0 = \mathrm{RCA}_0 + \mathrm{WKL}$.

The acronyms CA and RCA stand for *comprehension axiom* and *recursive comprehension axiom* respectively. Historically, theories in $\mathscr{L}_{\mathrm{II}}$ mostly came with induction for all $\mathscr{L}_{\mathrm{II}}$-formulas. When this is not the case, a subscript 0 was added. Since $\mathscr{L}_{\mathrm{II}}$-theories with full induction are increasing rare nowadays, many people have started using the unsubscripted names for other things.

Example 4.4. (1) $(\mathbb{N}, \mathcal{P}(\mathbb{N})) \models \mathrm{WKL}_0$.

(2) $(\mathbb{N}, \Delta_1\text{-Def}(\mathbb{N})) \models \mathrm{RCA}_0$, essentially because one can unravel each Δ_1-definable oracle into a program using Corollary 1.7. The same proof actually shows $(M, \Delta_1\text{-Def}(M)) \models \mathrm{RCA}_0$ for all $M \models \mathrm{I}\Sigma_1$.

We will do some coding inside these theories. So it is good to know in advance that coding works as expected there.

Proposition 4.5. $\mathrm{I}\Sigma_1 \vdash \exp$.

Proof. Using Fact 2.4, prove $\forall x\ \exists y\ (y = 2^x)$ by Σ_1-induction on x. □

4.2 The Arithmetized Completeness Theorem

Theorem 3.5 says that the coded subsets in every proper exponential cut satisfy the Weak König Lemma. We will prove a partial converse to this, that the first-order part of every model of WKL_0 is a proper exponential cut of some model of $\mathrm{I}\Delta_0 + \exp$.

For this, we need a method of building *end extensions* of a model of arithmetic M, i.e., extensions $K \supseteq_{\mathrm{e}} M$. The method we choose here is called the *Arithmetized Completeness Theorem.* This theorem is essentially a formalization of Gödel's Completeness Theorem for first-order logic within arithmetic. It is a surprisingly powerful method of constructing new models of arithmetic from existing ones.

For the formalization, we assume a well-behaved Gödel numbering $\ulcorner\cdot\urcorner$ of terms, formulas, sentences, proofs, etc. in which all the syntactical operations, such as substitution, are Δ_0-definable, and all the usual properties are provable in $\mathrm{I}\Delta_0 + \exp$. Often, we will identify a syntactical object with its Gödel number.

Definition. Given a theory T that is Δ_0-definable in $\mathbb{N}$, fix a Δ_0-formula $T\text{-Proof}(p,\theta)$ which expresses 'p is a proof of θ from T', and let

$$\mathrm{Con}(T) = \neg\exists p \ \ T\text{-Proof}(p,\bot).$$

Here $\bot$ denotes the logical constant for falsity. We sometimes view $T\text{-Proof}(p,\theta)$ and $\mathrm{Con}(T)$ as $\mathscr{L}_{\mathrm{II}}$-formulas in which T is a free set variable.

Remark 4.6. Notice if T is Δ_0-definable, then $\mathrm{Con}(T) \in \Pi_1$. When viewed as $\mathscr{L}_{\mathrm{II}}$-formulas, we know $T\text{-Proof}(p,\theta) \in \Delta^0_0$ and $\mathrm{Con}(T) \in \Pi^0_1$.

Lemma 4.7. RCA_0 proves the *Compactness Theorem* in the form

$$\forall T \ \big(\forall t \ \big(\mathrm{Ack}(t) \subseteq T \to \mathrm{Con}(\mathrm{Ack}(t))\big) \to \mathrm{Con}(T)\big).$$

Proof. Follow the usual proof. Notice by Δ_0-separation from Theorem 2.7, the set of assumptions used in a proof is always coded. □

Recall a theory H is *Henkinized* if for every formula θ, there is a constant symbol c such that

$$H \vdash \exists x \ \theta(x) \to \theta(c). \tag{$*$}$$

Sentences of the form above are sometimes called *Henkin axioms.* From a consistent Henkinized theory H, we 'read off' a *term model* K as follows.

- If c is a constant symbol, then $[c] = \{d : H \vdash c = d\}$.
- $K = \{[c] : c \text{ is a constant symbol}\}$.
- If c is a constant symbol, then $c^K = [c]$.
- If f is a function symbol and $[c] \in K$, then $f^K([c]) = [f(c)]$.
- If R is a relation symbol and $[c] \in K$, then $R^K([c])$ exactly when $R(c) \in H$.

One can verify that the term model originated from a consistent Henkinized theory H actually satisfies H. These enable us to rephrase the Completeness Theorem in terms of Henkinized theories.

Gödel's Completeness Theorem. Every consistent theory has a complete consistent Henkinized extension.

This reformulation can be readily formalized in $\mathscr{L}_{\mathrm{II}}$.

Theorem 4.8 (Simpson [6])**.** The following are equivalent over RCA_0.

(a) WKL.

(b) Gödel's Completeness Theorem.

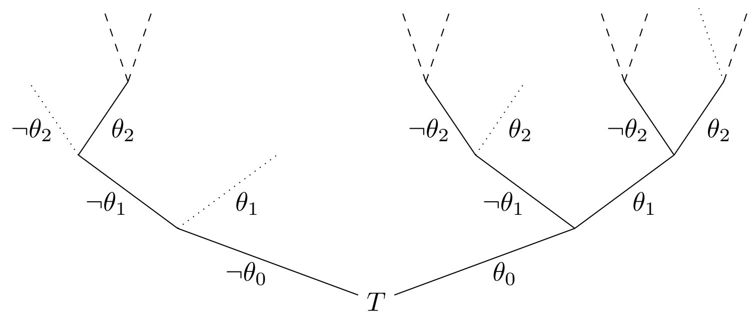

Figure 4.1: The tree of completions of a theory T

Proof sketch. Consider (a) $\Rightarrow$ (b). Henkinization can be achieved by adding sufficiently many Henkin axioms in a suitably expanded language. So we concentrate on completing the given consistent theory T. Let (θ_i) be an enumeration of all sentences. Denote

$$\Theta(\sigma) = \{\neg\theta_i : i < \operatorname{len}\sigma \wedge i \notin \operatorname{Ack}(\sigma)\} \cup \{\theta_i : i < \operatorname{len}\sigma \wedge i \in \operatorname{Ack}(\sigma)\}.$$

By Δ^0_1-CA, the set

$$\{\sigma : \exists p{<}\operatorname{len}\sigma\ (T + \Theta(\sigma))\text{-Proof}(p, \bot)\}$$

exists. Notice it is a binary tree. It is unbounded because if Ξ is a consistent set of sentences and θ is a formula, then either $\Xi \cup \{\theta\}$ or $\Xi \cup \{\neg\theta\}$ is consistent. One can verify that every unbounded branch in this tree corresponds to a complete consistent extension of T. So we are done by WKL.

We omit the proof of (b) $\Rightarrow$ (a) because it will not be used here. You may find it in Simpson's book [6, Theorem IV.3.3]. □

A side remark is that this theorem provides one way to separate WKL_0 from RCA_0.

Corollary 4.9. $(\mathbb{N}, \Delta_1\text{-Def}(\mathbb{N})) \not\models \mathrm{WKL}$.

Proof. Being satisfied in $\mathbb{N}$, we know PA is consistent. So $\mathbb{N} \models \mathrm{Con}(\mathrm{PA})$. However, Gödel's Incompleteness Theorem implies that no recursive extension of PA can be both consistent and complete. By Corollary 1.7, this means no complete consistent extension of PA is Δ_1-definable in $\mathbb{N}$. Therefore, our version of Gödel's Completeness Theorem fails in $(\mathbb{N}, \Delta_1\text{-Def}(\mathbb{N}))$. Theorem 4.8 then gives us the required failure of WKL. □

Notice Theorem 4.8 is entirely *internal* to the model of arithmetic in question. The major furtherance in the Arithmetized Completeness Theorem is in observing that, when viewed *externally*, the complete consistent Henkinized extension constructed using Theorem 4.8 gives rise to an *end* extension of the first-order part of the ground model.

Recall that if K is a structure for a language $\mathscr{L}$, then the *elementary diagram* of K, denoted $\mathrm{ElemDiag}(K)$, is defined by

$$\mathrm{ElemDiag}(K) = \{\theta(\bar{c}) : M \models \theta(\bar{c}) \text{ where } \theta \in \mathscr{L} \text{ and } \bar{c} \in K\}.$$

Definition. Let $\mathscr{L}$ be a language in $(M, \mathscr{X}) \models \mathrm{RCA}_0$, Then the non-logical symbols in $\mathscr{L}$ with standard arities constitute a language in the real world, which we denote by $\mathrm{Std}(\mathscr{L})$. Terms and formulas in $\mathrm{Std}(\mathscr{L})$ can be considered terms and formulas in $\mathscr{L}$ within $(M, \mathscr{X})$, but not necessarily the other way round. If T is a set of $\mathscr{L}$-formulas in $(M, \mathscr{X})$, then the set of all $\mathrm{Std}(\mathscr{L})$-formulas in T is denoted $\mathrm{Std}(T)$.

Arithmetized Completeness Theorem (ACT)**.** Let $(M, \mathscr{X}) \models \mathrm{WKL}_0$ and $T \in \mathscr{X}$ be a theory extending PA^-. If $(M, \mathscr{X}) \models \mathrm{Con}(T)$, then there exist $K \supseteq_{\mathrm{e}} M$ and $H \in \mathscr{X}$ such that

(a) $(M, \mathscr{X}) \models$ 'H is a complete consistent Henkinized theory extending T'; and

(b) $\mathrm{ElemDiag}(K) \subseteq H$.

In particular, such $K \models \mathrm{Std}(T)$.

Proof. Apply Theorem 4.8 to find $H \in \mathscr{X}$ satisfying (a). We know $\mathrm{Std}(H)$ is complete, consistent, and Henkinized in the real world because H is so in $(M, \mathscr{X})$. Let K be the term model of $\mathrm{Std}(H)$. Then the usual argument shows $\mathrm{ElemDiag}(K) = \mathrm{Std}(H) \subseteq H$. It remains to verify that $M \subseteq_{\mathrm{e}} K$.

Let $\mathscr{L}$ be the language of H. Notice $\mathscr{L} \supseteq \mathscr{L}_{\mathrm{A}}$ because $T \supseteq \mathrm{PA}^-$. So the language $\mathscr{L}$ in $(M, \mathscr{X})$ contains the term

$$\underline{a} = 0 + \underbrace{1 + 1 + \cdots + 1}_{a\text{-many 1's}}$$

for every $a \in M$. Exploiting the fact that H is Henkinized, find a constant symbol c_a in $\mathscr{L}$ for which $H \vdash c_a = \underline{a}$ for each $a \in M$. Abbreviating $\exists p\ \ \mathrm{PA}^-\text{-Proof}(p, \theta)$ as $\mathrm{PA}^-\text{-Prov}(\theta)$, one can verify using $\mathrm{I}\Sigma^0_1$ that $(M, \mathscr{X})$ satisfies

(1) $\mathrm{PA}^-\text{-Prov}(\ulcorner 0 = \underline{0} \wedge 1 = \underline{1} \urcorner)$;

(2) $\forall a, b\ \ \mathrm{PA}^-\text{-Prov}(\ulcorner \underline{a} + \underline{b} = \underline{a+b} \wedge \underline{a} \times \underline{b} = \underline{a \times b} \urcorner)$;

(3) $\forall a, b\ \big(a < b \to \mathrm{PA}^-\text{-Prov}(\ulcorner \underline{a} < \underline{b} \urcorner)\big)$;

(4) $\forall a, b\ \big(a \neq b \to \mathrm{PA}^-\text{-Prov}(\ulcorner \underline{a} \neq \underline{b} \urcorner)\big)$; and

(5) $\forall a\ \ \mathrm{PA}^-\text{-Prov}\big(\ulcorner \forall x {<} \underline{a+1}\ (x = \underline{0} \vee x = \underline{1} \vee \cdots \vee x = \underline{a}) \urcorner\big)$.

Therefore H contains

(1) $0 = c_0 \wedge 1 = c_1$;

(2) $c_a + c_b = c_{a+b} \wedge c_a \times c_b = c_{a \times b}$ for all $a, b \in M$;

(3) $c_a < c_b$ for all $a, b \in M$ such that $a < b$;

(4) $c_a \neq c_b$ for all distinct $a, b \in M$; and

(5) $\forall x {<} c_{a+1}\ (x = c_0 \vee x = c_1 \vee \cdots \vee x = c_a)$ for all $a \in M$.

The sentences in (1)–(4) above say that the map $a \mapsto c_a^K$ is an embedding $M \to K$. So we may regard $M \subseteq K$. The sentences in (5) tell us $M \subseteq_{\mathrm{e}} K$. To see this, let c be a constant symbol in $\mathrm{Std}(\mathscr{L})$ and $a \in M$ such that $[c] < c^K_{a+1} = [c_{a+1}]$. Then $\ulcorner c < c_{a+1} \urcorner \in H$. Applying (5), one obtains $i \leqslant a$ in M such that $\ulcorner c = c_i \urcorner \in H$. As $c = c_i$ is a $\mathrm{Std}(\mathscr{L})$-sentence in the real world, we know actually $\ulcorner c = c_i \urcorner \in \mathrm{Std}(H)$, and so $[c] = [c_i] = c_i^K$. □

In view of the Arithmetized Completeness Theorem, if we want to end extend the first-order part of $(M, \mathscr{X}) \models \mathrm{WKL}_0$ to a model of some theory T, then it suffices to show that $(M, \mathscr{X}) \models \mathrm{Con}(T^*)$ for some $T^* \in \mathscr{X}$ for which $\mathrm{Std}(T^*) \supseteq T$. So to get the promised converse to Theorem 3.5, all we need is an appropriate consistency condition.

Fact 4.10. $\mathrm{I}\Sigma_1 \vdash \mathrm{Con}(\mathrm{I}\Delta_0 + \exp)$.

Proof. It involves eliminating cuts in proofs, and is thus outside the scope of this course. Please consult Theorem II.8.11 in Simpson [6] for the details. □

Corollary 4.11. Let $(M, \mathscr{X}) \models \mathrm{WKL}_0$. Then there is $K \supsetneq_{\mathrm{e}} M$ such that $K \models \mathrm{I}\Delta_0 + \exp$.

Proof. Consider

$$\mathrm{I}\Delta_0 + \exp + \{c > \underline{a} : a \in M\},$$

where c is a new constant symbol, and $\underline{a}$ is as defined in the proof of the Arithmetized Completeness Theorem. This theory is in $(M, \mathscr{X})$ because of Δ^0_1-CA. It is consistent within $(M, \mathscr{X})$ by the Compactness Theorem in Lemma 4.7. An application of the Arithmetized Completeness Theorem gives $K \supseteq_{\mathrm{e}} M$ that satisfies $\mathrm{I}\Delta_0 + \exp$. This extension K of M is proper because of c^K. □

There is a caveat in the proof above: the definition of $\mathrm{I}\Delta_0 + \exp$ in M is given by an $\mathscr{L}_{\mathrm{A}}$-formula via Corollary 1.7, and the models M and $\mathbb{N}$ may disagree on this formula. So it may happen that an axiom of the real $\mathrm{I}\Delta_0 + \exp$ is actually not an element of the M-version of $\mathrm{I}\Delta + \exp$. Fortunately, this is not possible because $\mathrm{I}\Delta_0 + \exp$ can be defined by a Δ_0-formula.

Proposition 4.12. Let $M, K \models \mathrm{PA}^-$ such that $M \subseteq_\mathrm{e} K$. Then

$$M \models \xi(\bar{c}) \quad \Leftrightarrow \quad K \models \xi(\bar{c})$$

for all $\bar{c} \in M$ and $\xi \in \Delta_0$.

Proof sketch. Induction on ξ. For example, if $M \models \forall x{<}t(\bar{c})\ \xi(x, \bar{c})$ where t is some $\mathscr{L}_\mathrm{A}$-term and ξ is a formula that satisfies the proposition, then $K \models \forall x{<}t(\bar{c})\ \xi(x, \bar{c})$ too by the induction hypothesis, because $t(\bar{c}) \in M$ and so every $x < t(\bar{c})$ in K must also be in M. □

Further exercises

Corollary 4.11 tells us that given any $(M, \mathscr{X}) \models \mathrm{WKL}_0$, we can find $K \models \mathrm{I}\Delta_0 + \exp$ in which $M \subseteq_\mathrm{e} K$. It says nothing about the relationship between $\mathscr{X}$ and $\mathrm{Cod}(K/M)$, although one would expect them to be somehow related in view of Theorem 3.5. These exercises will show that this suspicion is correct, at least for countable models.

Theorem 4.13 (Scott [5]; Tanaka [8])**.** Let $(M, \mathscr{X})$ be a countable model of WKL_0. Then there is $K \supsetneq_\mathrm{e} M$ such that $K \models \mathrm{I}\Delta_0 + \exp$ and $\mathrm{Cod}(K/M) = \mathscr{X}$.

Proof. This is taken from Enayat [1]. Follow the proof of Corollary 4.11 to find $H_0 \in \mathscr{X}$ such that $(M, \mathscr{X})$ believes H_0 to be a complete consistent Henkinized extension of $\mathrm{I}\Delta_0 + \exp$. By recursion, we will build a sequence $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ of theories in $\mathscr{X}$ in increasingly big languages. Each of these theories will be complete, consistent, and Henkinized from the point of view of $(M, \mathscr{X})$.

Enumerate $\mathscr{X}$ as $(D_n)_{n\in\mathbb{N}}$ using countability. Suppose we already found H_n satisfying the inductive conditions. Let d_n be a new constant symbol. Remember how the $\underline{a}$'s were defined in the proof of the Arithmetized Completeness Theorem.

(a) Show that $(M, \mathscr{X})$ models the consistency of

$$H_n \cup \{\underline{a} \in \mathrm{Ack}(d_n) : a \in D_n\} \cup \{\underline{a} \notin \mathrm{Ack}(d_n) : a \in M \setminus D_n\}.$$

Use the Arithmetized Completeness Theorem to find $H_{n+1} \in \mathscr{X}$ that $(M, \mathscr{X})$ believes to be complete, consistent, Henkinized, and extends the theory displayed above.

Set $H = \bigcup_{n\in\mathbb{N}} H_n$.

(b) Explain why $\mathrm{Std}(H)$ is a complete consistent Henkinized extension of $\mathrm{I}\Delta_0 + \exp$.

Let K be the term model of $\mathrm{Std}(H)$.

(c) Show that M can be considered a *proper* cut of K.

(d) Explain why $\mathscr{X} \subseteq \mathrm{Cod}(K/M)$.

(e) Use Δ^0_1-CA in $(M, \mathscr{X})$, or otherwise, to show that $\mathrm{Cod}(K/M) \subseteq \mathscr{X}$. □

Further reading

- Theorem 4.8 is an example of a theorem from *Reverse Mathematics*, a subject in which the exact axioms needed to prove various mathematical theorems are determined. Simpson's book [6] is the definitive reference for Reverse Mathematics. You may find in his Theorem IV.3.3 that if the Compactness Theorem is formulated as

  if every coded subset of a theory T has a model, then T has a model,

  then in contrary to Lemma 4.7 above, it is equivalent to WKL over RCA_0.

- The Arithmetized Completeness Theorem is arguably the most powerful method in building end extensions of models of arithmetic. In particular, it is one of the few techniques that works even for uncountable models. See Smoryński's notes [7] for more applications. Traditionally, the Arithmetized Completeness Theorem is formulated within first-order arithmetic. Our approach via second-order arithmetic has the advantage of allowing easier iterations, as demonstrated in the Further exercises.

- There is a gap between WKL in Theorem 3.5 and WKL_0 in Theorem 4.13. These two sides are known to meet at a theory called WKL_0^*, at least when the exp condition on the bigger model is dropped. The theory WKL_0^* is defined to be $\mathrm{PA}^- + \mathrm{I}\Delta_0^0 + \exp + \Delta_1^0\text{-CA}$.

  Theorem 4.14 (Scott [5]). Let $M \models \mathrm{I}\Delta_0$ and I be a proper exponential cut of M. Then $(M, I) \models \mathrm{WKL}_0^*$.

  Theorem 4.15 (Enayat–Wong [2]). Let $(M, \mathscr{X})$ be a countable model of WKL_0^*. Then there exists $K \supsetneq_{\mathrm{e}} M$ such that $K \models \mathrm{I}\Delta_0$ and $\mathrm{Cod}(K/M) = \mathscr{X}$.

  It is not known whether this theorem would remain true if the countability condition is dropped. The special case when the first-order part is standard is perhaps one of the most well-known open questions in the model theory of arithmetic.

  Question 4.16 (Scott [5]). Is it true that every $(\mathbb{N}, \mathscr{X}) \models \mathrm{WKL}_0$ realizes as $(\mathbb{N}, \mathrm{Cod}(M/\mathbb{N}))$ for some $M \models \mathrm{I}\Delta_0$?

  Knight and Nadel [3] showed that Scott's question has a positive answer if $\mathscr{X}$ is additionally required to have size at most $\aleph_1$. Therefore, in the situation when the Continuum Hypothesis holds, we already have a full answer.

References

[1] Ali Enayat. A new proof of Tanaka's theorem. In Patrick Cégielski, Charalampos Cornaros, and Constantinos Dimitracopoulos, editors. *New Studies in Weak Arithmetics*, volume 211 of *CSLI Lecture Notes*, pages 93–102. CSLI Publications, Stanford, September 2013.

[2] Ali Enayat and Tin Lok Wong. Model theory of WKL_0^*. Forthcoming.

[3] Julia Knight and Mark Nadel. Models of arithmetic and closed ideals. *The Journal of Symbolic Logic*, 47(4):833–840, December 1982.

[4] Roman Kossak and James H. Schmerl. *The Structure of Models of Peano Arithmetic*, volume 50 of *Oxford Logic Guides*. Clarendon Press, Oxford, 2006.

[5] Dana Scott. Algebras of sets binumerable in complete extensions of arithmetic. In Jacob C.E. Dekker, editor. *Recursive Function Theory*, volume V of *Proceedings of Symposia in Pure Mathematics*, pages 117–121. American Mathematical Society, Providence, Rhode Island, 1962.

[6] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*, volume 1 of *Perspectives in Logic*. Cambridge University Press, Cambridge, second edition, 2009.

[7] Craig Smoryński. Lectures on nonstandard models of arithmetic. In Gabriele Lolli, Giuseppe Longo, and Annalisa Marcja, editors. *Logic Colloquium '82*, volume 112 of *Studies in Logic and the Foundations of Mathematics*, pages 1–70. North-Holland Publishing Company, Amsterdam, 1984.

[8] Kazuyuki Tanaka. The self-embedding theorem of WKL_0 and a non-standard method. *Annals of Pure and Applied Logic*, 84(1):41–49, March 1997.