MODEL THEORY OF ARITHMETIC

Lecture 6: End and cofinal extensions

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> A relatively neglected aspect of the study of nonstandard models of arithmetic is the study of their cofinal extensions. These extensions certainly do not present themselves to the intuition as readily as do their more popular cousins the end extensions; but they are not exactly shrouded in mystery or unnatural objects of study either. They are equal partners with end extensions in the construction of general extensions of models; they offer both special advantages and disadvantages worthy of our interest; and, occasionally, they are useful in understanding the generally more simply behaved end extensions.
>
> Craig Smoryński [17]

Recall $\mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$ for every $n \in \mathbb{N}$ from Theorem 2.10. The aim of this lecture is to show that actually these theories are not very far apart.

Theorem 6.1 (Harvey Friedman, Jeff Paris [14], independently)**.** Let $n \in \mathbb{N}$. Then $\mathrm{B}\Sigma_{n+1}$ is Π_{n+2}*-conservative* over $\mathrm{I}\Sigma_n$, i.e., for every $\sigma \in \Pi_{n+2}$,

$$\mathrm{B}\Sigma_{n+1} \vdash \sigma \quad \Rightarrow \quad \mathrm{I}\Sigma_n \vdash \sigma.$$

The particular case when $\sigma = \bot$ is essentially the equiconsistency between $\mathrm{B}\Sigma_{n+1}$ and $\mathrm{I}\Sigma_n$. However, this conservation theorem tells us much more than just equiconsistency. For example, establishing the equiconsistency between $\mathrm{B}\Sigma_{n+1}$ and $\mathrm{I}\Sigma_n$ in ZFC is trivial, because ZFC proves the consistencies of both $\mathrm{B}\Sigma_{n+1}$ and $\mathrm{I}\Sigma_n$. Contrarily, this conservation result, even established within ZFC, really has non-trivial content.

We know that $\mathrm{I}\Sigma_{n+1}$ is never Π_1-conservative over the corresponding $\mathrm{B}\Sigma_{n+1}$ by Gödel's Second Incompleteness Theorem, because $\mathrm{I}\Sigma_{n+1} \vdash \mathrm{Con}(\mathrm{B}\Sigma_{n+1})$. A proof of this can be found in Section I.4 of the Hájek–Pudlák book [10].

The proof of Theorem 6.1 we present here is similar to those in Clote–Hájek–Paris [5, Section 3] and Kaye [11, proof of Theorem 3.2], using *end* and *cofinal* extensions. Recall

$$\begin{array}{lll} K \supseteq_{\mathrm{e}} M & \text{means} & \forall k{\in}K \setminus M\ \forall m{\in}M\ m \leqslant k, \\ K \supseteq_{\mathrm{cf}} M & \text{means} & \forall k{\in}K \setminus M\ \exists m{\in}M\ k \leqslant m. \end{array}$$

We split into two threads, which will merge in the end to give the conservation theorem.

6.1 The strength of end extensions

We may define $\mathrm{B}\Pi_n$ in a way analogous to the definition of $\mathrm{B}\Sigma_n$, but as mentioned in Proposition 3.8, we get no new theory out of this. We prove the proposition in full here.

Proposition 6.2. $\mathrm{B}\Sigma_{n+1}$ and $\mathrm{B}\Pi_n$ are equivalent for all $n \in \mathbb{N}$.

Proof. If $\forall x{<}a\ \exists y\ \exists v\ \psi(v, x, y, \bar{z})$, where $\psi \in \Pi_n$, then applying $\mathrm{B}\Pi_n$ gives

$$\exists b\ \forall x{<}a\ \exists y, v{<}b\ \psi(v, x, y, \bar{z}),$$

so that $\exists b\ \forall x{<}a\ \exists y{<}b\ \exists v\ \psi(v, x, y, \bar{z})$ in particular. □

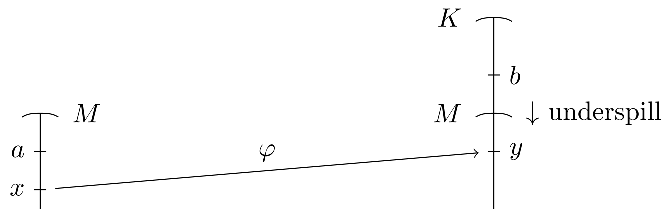

Figure 6.1: How end extensions imply collection

We need also an upside-down version of overspill.

Underspill. Let $n \in \mathbb{N}$ and $I \subsetneq_{\mathrm{e}} M \models \mathrm{I}\Sigma_n$. If $\theta \in \Pi_n(M)$ such that

$$M \models \theta(x) \qquad \text{for all } x \in M \setminus I,$$

then $M \models \theta(x)$ for arbitrarily large $x \in I$.

Proof. Otherwise $x > b \wedge \theta(x)$ defines $M \setminus I$ for some $b \in I$, contradicting $\mathrm{L}\Pi_n$. □

The collection schemes are intimately connected with end extensions. In general, if a model of arithmetic has a proper end extension, then it satisfies some collection. The amount of collection it satisfies depends on what theory the extension has, and how elementary the extension is.

Definition. Let M, K be $\mathscr{L}_{\mathrm{A}}$-structures and $n \in \mathbb{N}$. Then M is a Σ_n*-elementary*, or simply n*-elementary*, substructure of K, written $M \preccurlyeq_n K$, if $M \subseteq K$ and for all $\theta \in \Sigma_n$ and all $\bar{c} \in M$,

$$M \models \theta(\bar{c}) \quad \Leftrightarrow \quad K \models \theta(\bar{c}).$$

In this case, we informally say that such $\theta(\bar{c})$'s *transfer* between M and K.

Notice requiring $\theta \in \Pi_n$ instead does not change the notion defined.

Theorem 6.3 (Paris–Kirby [16], Adamowicz–Clote–Wilkie [4])**.** Fix $n \in \mathbb{N}$. Let $M \models \mathrm{PA}^-$ and $K \succneqq_{n,\mathrm{e}} M$ satisfying $\mathrm{I}\Sigma_n$. Then $M \models \mathrm{B}\Sigma_{n+1}$.

Proof. Notice $M \models \mathrm{I}\Delta_0$ by Corollary 5.1. In view of Proposition 6.2, it suffices to show $M \models \mathrm{B}\Pi_n$. Let $a \in M$ and $\varphi \in \Pi_n(M)$ such that $M \models \forall x{<}a\ \exists y\ \varphi(x, y)$. Consider

$$\{b \in K : K \models \underbrace{\forall x{<}a\ \exists y{<}b\ \underbrace{\varphi(x, y)}_{\Pi_n}}_{\Pi_n \text{ over } K \models \mathrm{B}\Sigma_n}\}.$$

It includes $K \setminus M$ because $M \preccurlyeq_{n,\mathrm{e}} K$. So by Π_n-underspill, we find $b \in M$ such that

$$K \models \forall x{<}a\ \exists y{<}b\ \varphi(x, y).$$

This transfers down to M since $M \preccurlyeq_{n,\mathrm{e}} K$. □

Adamowicz, Clote and Wilkie [4] actually showed that if $n \geqslant 1$, then one can weaken $\mathrm{I}\Sigma_n$ to $\mathrm{I}\Sigma_{n-1}$ in the theorem above.

6.2 The Splitting Theorem

We need a hierarchical version of the Tarski–Vaught Test [18]. The significance of this test is that it allows us to talk about elementarity without reference to what is true in the smaller model. This is especially helpful when we do not know much about the smaller model, for example, when it is still being constructed.

Tarski–Vaught Test. Let M, K be $\mathscr{L}_{\mathrm{A}}$-structures, and suppose $M \preccurlyeq_0 K$. Then the following are equivalent for all $n \in \mathbb{N}$.

(a) $M \preccurlyeq_{n+1} K$.

(b) For every $\eta(\bar{x}) \in \Pi_n(M)$, if $K \models \exists \bar{x}\ \eta(\bar{x})$, then $K \models \eta(\bar{c})$ for some $\bar{c} \in M$.

Proof. The implication (a) $\Rightarrow$ (b) is straightforward. We prove the converse implication (b) $\Rightarrow$ (a) by strong induction on n.

Let $n \in \mathbb{N}$ for which (b) is true, and (b) $\Rightarrow$ (a) for all smaller indices. Then we know $M \preccurlyeq_n K$, either from the assumption $M \preccurlyeq_0 K$ when $n = 0$, or from the induction hypothesis when $n > 0$. Consider the $\Sigma_{n+1}(M)$-formula $\exists \bar{x}\ \eta(\bar{x})$, where $\eta \in \Pi_n(M)$. Clearly $M \models \exists \bar{x}\ \eta(\bar{x})$ implies $K \models \exists \bar{x}\ \eta(\bar{x})$ since $M \preccurlyeq_n K$. If $K \models \exists \bar{x}\ \eta(\bar{x})$, then

$$\begin{array}{llll} & K \models \eta(\bar{c}) & \text{for some } \bar{c} \in M & \text{by (b)} \\ \therefore & M \models \eta(\bar{c}) & \text{for some } \bar{c} \in M & \text{since } M \preccurlyeq_n K \\ \therefore & M \models \exists \bar{x}\ \eta(\bar{x}). & & \end{array}$$

□

Another fact that we need is an alternative axiomatization of the induction schemes in terms of *strong collection*. Recall that collection says all functions with bounded domains have bounded images. Strong collection says that all *partial* functions with bounded domains, not only total ones, have bounded images. This gives yet another way of showing $\mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$ for every $n \in \mathbb{N}$.

Theorem 6.4 (Harvey Friedman)**.** The following are equivalent over $\mathrm{I}\Delta_0$ for all $n \in \mathbb{N}$.

(a) $\mathrm{I}\Sigma_{n+1}$.

(b) *Strong* Π_n*-collection*: for every $\eta \in \Pi_n$,

$$\forall \bar{z}\ \forall a\ \exists b\ \forall x{<}a\ \big(\exists y\ \eta(x, y, \bar{z}) \to \exists y{<}b\ \eta(x, y, \bar{z})\big).$$

Proof. For (a) $\Rightarrow$ (b), use Σ_{n+1}-separation from Theorem 2.7 to find c such that

$$\mathrm{Ack}(c) = \{x < a : \exists y\ \eta(x, y, \bar{z})\}.$$

Recall $\mathrm{I}\Sigma_{n+1} \vdash \mathrm{B}\Sigma_{n+1}$ from Theorem 2.2. So

$$\exists b\ \forall x{<}a\ \exists y{<}b\ \underbrace{\big(\underbrace{x \in \mathrm{Ack}(c)}_{\Delta_0} \to \underbrace{\eta(x, y, \bar{z})}_{\Pi_n}\big)}_{\Pi_n}.$$

We show (b) $\Rightarrow$ (a) by strong induction on n. Let $n \in \mathbb{N}$ such that strong Π_n-collection holds, and (b) $\Rightarrow$ (a) for all smaller indices. Then we get $\mathrm{I}\Sigma_n$, either from the base theory $\mathrm{I}\Delta_0$ when $n = 0$, or from the induction hypothesis when $n > 0$. As in the proof of Theorem 5.9, it suffices to show that every nonempty bounded Σ_{n+1}-definable set has a minimum. Fix a and consider the Σ_{n+1}-formula $\exists y\ \eta(x, y, \bar{z})$, where $\eta \in \Pi_n$. Strong Π_n-collection gives b such that

$$\{x < a : \exists y\ \eta(x, y, \bar{z})\} = \{x < a : \underbrace{\exists y{<}b\ \underbrace{\eta(x, y, \bar{z})}_{\Pi_n}}_{\Pi_n \text{ over } \mathrm{B}\Sigma_n}\}.$$

This set is Π_n-definable because either $n = 0$ and $\mathrm{B}\Sigma_n$ is not needed, or $n > 0$ and we have $\mathrm{B}\Sigma_n$ from $\mathrm{I}\Sigma_n$. So provided it is nonempty, it must have a minimum by $\mathrm{L}\Pi_n$. □

Remark 6.5. Similar to Proposition 6.2, strong Σ_{n+1}-collection, appropriately defined, is the same as strong Π_n-collection for every $n \in \mathbb{N}$.

We are now ready for the *Splitting Theorem*, which says that every extension splits into a cofinal extension followed by an end extension. Clearly, to split an extension in such a way, the middle model must consist of the downward closure of the ground model.

Definition. If $S \subseteq M \models \mathrm{PA}^-$, then

$$\sup_M S = \{x \in M : x \leqslant s \text{ for some } s \in S\}.$$

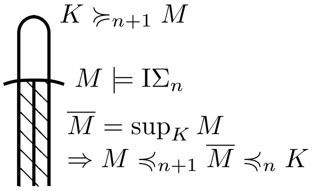

Figure 6.2: The Splitting Theorem

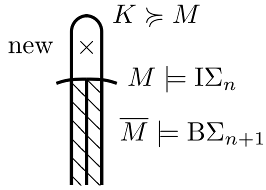

Figure 6.3: Constructing cofinal extensions using the Splitting Theorem

More importantly, some elementarity is preserved in such splits. This should be surprising because the supremum of a model is an entirely order-theoretic notion, and it is not clear at first glance why it should have any bearing on the arithmetic structure.

Splitting Theorem (essentially Kaye [11])**.** Fix $n \in \mathbb{N}$. Let $M \models \mathrm{I}\Sigma_n$ and $K \succcurlyeq_{n+1} M$. Then $M \preccurlyeq_{n+1,\mathrm{cf}} \overline{M} \preccurlyeq_{n,\mathrm{e}} K$, where $\overline{M} = \sup_K M$.

Proof. We follow the proof from Enayat–Mohsenipour [7].

We first show the n-elementarity between $\overline{M}$ and K. Note $\overline{M} \subseteq_{\mathrm{e}} K$. So $\overline{M} \preccurlyeq_0 K$ by Proposition 4.12. Thus we are already done if $n = 0$. Now, suppose $n = m + 1$. Then the Tarski–Vaught Test applies. Let $\eta(x, y) \in \Pi_m$ and $c \in \overline{M}$ such that $K \models \exists y\ \eta(c, y)$. Take $a \in M$ strictly above c, which is possible since $M \subseteq_{\mathrm{cf}} \overline{M}$. Using strong Π_m-collection, find $b \in M$ such that

$$M \models \underbrace{\forall x{<}a\ \big(\underbrace{\exists y\ \underbrace{\eta(x,y)}_{\Pi_m}}_{\Sigma_{m+1}} \to \underbrace{\exists y{<}b\ \underbrace{\eta(x,y)}_{\Pi_m}}_{\Sigma_{m+1}}\big)}_{\Pi_{m+2}}.$$

This transfers to K by $(m + 2)$-elementarity, making $K \models \exists y{<}b\ \eta(c, y)$. Any witness to this must be in $\overline{M}$ because $b \in M$.

Let us move on to the $(n + 1)$-elementarity between M and $\overline{M}$. First, we know $M \preccurlyeq_n \overline{M}$ since $M \preccurlyeq_{n+1} K \succcurlyeq_n \overline{M}$. Let $\eta(x) \in \Pi_n(M)$. Then $M \models \exists x\ \eta(x)$ implies $\overline{M} \models \exists x\ \eta(x)$ because $M \preccurlyeq_n \overline{M}$. Conversely, if $\overline{M} \models \exists x\ \eta(x)$, then

$$\begin{array}{lll} & K \models \exists x\ \eta(x) & \text{since } \overline{M} \preccurlyeq_n K \\ \therefore & M \models \exists x\ \eta(x) & \text{since } M \preccurlyeq_{n+1} K. \end{array}$$
□

The Splitting Theorem tells us that end extensions and cofinal extensions are essentially the only interesting kinds of extensions for models of arithmetic, because every other extension factors into these.

There are a number of other splitting theorems in the model theory of arithmetic in the literature. We will meet one more in the Further exercises.

6.3 Cofinal extensions

We are only one step away from the conservation result. The stepping stone is, nevertheless, of independent model-theoretic interest. Its countable version first appeared in Paris [14], in which Harvey Friedman is reported to have discovered the same theorem independently.

Theorem 6.6. Let $n \in \mathbb{N}$ and $M \models \mathrm{I}\Sigma_n$. Then there is $\overline{M} \succcurlyeq_{n+1,\mathrm{cf}} M$ that satisfies $\mathrm{B}\Sigma_{n+1}$.

Proof. Use Compactness to find $K \succcurlyeq M$ such that $M \not\subseteq_{\mathrm{cf}} K$. Let $\overline{M} = \sup_K M$. The Splitting Theorem implies $\overline{M} \precnsim_{n,\mathrm{e}} K$ and $M \preccurlyeq_{n+1,\mathrm{cf}} \overline{M}$. So $\overline{M} \models \mathrm{B}\Sigma_{n+1}$ by Theorem 6.3. □

This proof actually shows the stronger fact that the supremum of a model of $\mathrm{I}\Sigma_n$ in *every* non-cofinal $(n + 1)$-elementary extension is a model of $\mathrm{B}\Sigma_{n+1}$.

Theorem 6.6 readily implies the Friedman–Paris conservation theorem, because if $\mathrm{B}\Sigma_{n+1}$ proves much more than what $\mathrm{I}\Sigma_n$ does, then there must be a model of $\mathrm{I}\Sigma_n$ which deviates from $\mathrm{B}\Sigma_{n+1}$ so much that it cannot fit inside any model of $\mathrm{B}\Sigma_{n+1}$.

Proof of Theorem 6.1. Let $\sigma \in \Pi_{n+2}$ such that $\mathrm{I}\Sigma_n \nvdash \sigma$. Pick $M \models \mathrm{I}\Sigma_n + \neg\sigma$. Find $\overline{M} \succcurlyeq_{n+1} M$ satisfying $\mathrm{B}\Sigma_{n+1}$ using Theorem 6.6. Then $\overline{M} \models \neg\sigma$ because $\neg\sigma \in \Sigma_{n+2}$. So $\mathrm{B}\Sigma_{n+1} \nvdash \sigma$. □

Further exercises

The MRDP Theorem, mentioned in the Further reading section of Lecture 1, says that every Σ_1-formula is uniformly equivalent to an existential $\mathscr{L}_\mathrm{A}$-formula in $\mathbb{N}$. A careful check reveals that its proof can actually be formalized within $\mathrm{I}\Delta_0 + \exp$. So we informally write $\mathrm{I}\Delta_0 + \exp \vdash \mathrm{MRDP}$.

Fact 6.7 (Gaifman–Dimitracopoulos [9])**.** For every Σ_1-formula $\theta(\bar{x})$, there exists an existential $\mathscr{L}_\mathrm{A}$-formula $\theta'(\bar{x})$ such that $\mathrm{I}\Delta_0 + \exp \vdash \forall \bar{x}\ \big(\theta(\bar{x}) \leftrightarrow \theta'(\bar{x})\big)$.

The aim of these exercises is to prove the Gaifman Splitting Theorem [8, 9] assuming MRDP holds in $\mathrm{I}\Delta_0 + \exp$. The main difference between our Splitting Theorem and Gaifman's is that his theorem does *not* start with any elementarity. See Kaye [12, Section 7.2] for a related discussion.

Gaifman Splitting Theorem. If $M, K \models \mathrm{PA}$ such that $M \subseteq K$, then $\overline{M} = \sup_K M \succcurlyeq M$.

(a) Recall $\Delta_0 \subseteq \Sigma_1 \cap \Pi_1$. Show that if $M, K \models \mathrm{I}\Delta_0 + \exp$ and $M \subseteq K$, then $M \preccurlyeq_0 K$.

(b) Let θ be an $\mathscr{L}_\mathrm{A}(M)$-formula and $M \subseteq_\mathrm{cf} \overline{M} \models \mathrm{PA}^-$. Suppose for every $a \in M$, there exists $b \in M$ such that $\overline{M} \models \forall x{<}a\ \exists y{<}b\ \theta(x, y)$. Explain why $\overline{M} \models \forall x\ \exists y\ \theta(x, y)$.

(c) Using (b), or otherwise, show that if $M \models \mathrm{PA}^- + \mathrm{Coll}(\Sigma_1)$ and $\overline{M} \succcurlyeq_{0,\mathrm{cf}} M$ satisfying PA^-, then $M \preccurlyeq_2 \overline{M}$.

(d) Fix $n \in \mathbb{N}$. Follow the steps below to show that if $M \models \mathrm{B}\Sigma_{n+1}$ and $\overline{M} \succcurlyeq_{n,\mathrm{cf}} M$, then $\overline{M} \succcurlyeq_{n+2} M$.

We are done if $n = 0$ by (c). So suppose $n \geqslant 1$. Let $\theta \in \Pi_n(M)$ such that $M \models \forall x\ \exists y\ \theta(x, y)$. Since $M \subseteq_\mathrm{cf} \overline{M}$, it suffices to show $\overline{M} \models \forall x{<}a\ \exists y\ \theta(x, y)$ for every $a \in M$. Pick $a \in M$. Define $f\colon \{x \in M : x < a\} \to M$ by setting

$$f(x) = \min\{y \in M : M \models \theta(x, y)\}.$$

(i) Explain why f is well-defined and is coded in M.

(ii) Show that the code of f in M also codes a total function $\{x \in \overline{M} : x < a\} \to \overline{M}$ in $\overline{M}$.

(iii) Explain why $\overline{M} \models \forall x{<}a\ \theta(x, f(x))$. Remember we do not know yet whether $\overline{M} \models \mathrm{B}\Sigma_n$.

(e) Combine (c) and (d) to show that if $M \models \mathrm{PA}$ and $\overline{M} \succcurlyeq_{0,\mathrm{cf}} M$ satisfying PA^-, then $\overline{M} \succcurlyeq M$.

(f) Deduce the Gaifman Splitting Theorem from (a) and (e). Observe that the bigger model actually does not need to satisfy full PA: having $\mathrm{I}\Delta_0 + \exp$ is sufficient.

Further comments

More about the Friedman–Paris conservation theorem

Many proofs of Theorem 6.1 are known. The first one by Paris [14] involves his hierarchy of cuts. Later he gave a simpler proof using definable ultrapowers [15]. Other model-theoretic proofs use Skolem hulls [12, Chapter 10] or Herbrand saturated models [1]. Several proof-theoretic proofs are known too [2, 3].

The Π_{n+2}-conservativity of $\mathrm{B}\Sigma_{n+1}$ over $\mathrm{I}\Sigma_n$ is actually provable in $\mathrm{I}\Delta_0 + \mathrm{supexp}$, where supexp is an axiom asserting the existence of

$$2^{2^{\cdot^{\cdot^{2}}}} \Big\}\ x\text{-many 2's}$$

for every x. See the paper by Clote, Hájek, and Paris [5] for more information.

Collection and end extensions

The theory $\mathrm{B}\Sigma_{n+1}$ we obtained in Theorem 6.3 is best possible. A lot is known about similar equivalences between end extendability and strength. These results can be summarized as follows.

Definition. Let $k, n \in \mathbb{N}$. Then $\Pi^*_{k+1}\text{-Cn}(\mathrm{I}\Sigma_n) = \{\forall\bar{x}\ \theta(\bar{x}) : \theta \in \langle\Sigma_k\rangle_\Delta \text{ and } \mathrm{I}\Sigma_n \vdash \forall\bar{x}\ \theta(\bar{x})\}$ and $\mathrm{Th}_{\mathrm{I}\Sigma_n}(k\text{-ecuts}) = \bigcap\{\mathrm{Th}(I) : I \precneqq_{k,\mathrm{e}} M \models \mathrm{I}\Sigma_n\}$, where $\langle\Sigma_k\rangle_\Delta$ denotes the closure of Σ_k under Boolean operations and bounded quantification.

Theorem 6.8. Let $n, k \in \mathbb{N}$. Then, modulo logical equivalence,

$$\mathrm{Th}_{\mathrm{I}\Sigma_n}(k\text{-ecuts}) = \begin{cases} \mathrm{B}\Sigma_{k+1} + \Pi^*_{k+1}\text{-Cn}(\mathrm{I}\Sigma_n), & \text{if } k \leqslant n+1; \\ \mathrm{B}\Sigma_k, & \text{if } k \geqslant n+2. \end{cases}$$

Moreover, every model of $\mathrm{Th}_{\mathrm{I}\Sigma_n}(k\text{-ecuts})$ is elementarily equivalent to some $I \precneqq_{k,\mathrm{e}} M \models \mathrm{I}\Sigma_n$.

The case when $k \geqslant n+2$ can be extracted from Paris–Kirby [16]. One direction of the case when $k \leqslant n+1$ is provided by our Theorem 6.3. The other direction can be proved using Theorem 8.5 and Theorem 12.4.

Notice Theorem 6.8 does *not* say that every model of $\mathrm{Th}_{\mathrm{I}\Sigma_n}(k\text{-ecuts})$ is a proper k-elementary cut of some model of $\mathrm{I}\Sigma_n$. Apparently, some extra saturation is needed to guarantee the existence of an end extension of the appropriate kind [6], but currently, the exact amount of saturation required is mostly unknown. Here we gather the key questions.

Question 6.9 (Paris [15, Problem 1])**.** Does every countable model of $\mathrm{B}\Sigma_1$ have a proper end extension satisfying $\mathrm{I}\Delta_0$?

Question 6.10 (Clote [4])**.** Let $n \in \mathbb{N}$. Does every model of $\mathrm{B}\Sigma_{n+2}$ have a proper $(n+2)$-elementary end extension satisfying $\mathrm{I}\Delta_0$?

Question 6.11 (Clote [4], after Matt Kaufmann)**.** Let $n \in \mathbb{N}$. Does every countable model of $\mathrm{B}\Sigma_{n+2}$ have a proper $(n+2)$-elementary end extension satisfying $\mathrm{B}\Sigma_{n+1}$?

Wilkie and Paris [20] showed that Questions 6.9 and 3.9 cannot both have a positive answer. Even the following variant of Question 6.11 seems to be open: is it true that for every $n \in \mathbb{N}$,

$$\bigcap\{\mathrm{Th}(I) : I \precneqq_{\mathrm{e},n+2} M \models \mathrm{B}\Sigma_{n+1}\} \subseteq \mathrm{B}\Sigma_{n+2}?$$

Formalizing the MRDP Theorem

There is a sense in which MRDP Theorem is equivalent to the Gaifman Splitting Theorem, see Exercise 7.6 in Kaye's book [3]. It is open whether Fact 6.7 remains true without exp.

Question 6.12 (Jeff Paris)**.** Does $\mathrm{I}\Delta_0 \vdash \mathrm{MRDP}$?

Wilkie [19] noticed that a positive answer to this question implies NP = co-NP.

References

[1] Jeremy Avigad. Saturated models of universal theories. *Annals of Pure and Applied Logic*, 118(3):219–234, December 2002.

[2] Lev D. Beklemishev. A proof-theoretic analysis of collection. *Archive for Mathematical Logic*, 37(5–6):275–296, July 1998.

[3] Samuel R. Buss. The witness function method and provably recursive functions of Peano arithmetic. In Dag Prawitz, Brian Skyrms, and Dag Westerståhl, editors. *Logic, Methodology and Philosophy of Science IX*, volume 134 of *Studies in Logic and the Foundations of Mathematics*, pages 29–68. North-Holland Publishing Company, Amsterdam, 1994.

[4] Peter G. Clote. Partition relations in arithmetic. In Carlos Augusto Di Prisco, editor. *Methods in Mathematical Logic*, volume 1130 of *Lecture Notes in Mathematics*, pages 32–68. Springer-Verlag, Berlin, 1985.

[5] Peter G. Clote, Petr Hájek, and Jeff B. Paris. On some formalized conservation results in arithmetic. *Archive for Mathematical Logic*, 30(4):201–218, 1990.

[6] Charalampos Cornaros and Constantinos Dimitracopoulos. On two problems concerning end-extensions. *Archive for Mathematical Logic*, 47(1):1–14, June 2008.

[7] Ali Enayat and Shahram Mohsenipour. Model theory of the regularity and reflection schemes. *Archive for Mathematical Logic*, 47(5):447–464, 2008.

[8] Haim Gaifman. A note on models and submodels of arithmetic. In Wilfrid Hodges, editor. *Conference in Mathematical Logic — London '70*, volume 255 of *Lecture Notes in Mathematics*, pages 128–144. Springer-Verlag, Berlin, 1972.

[9] Haim Gaifman and Constantinos Dimitracopoulos. Fragments of Peano's arithmetic and the MRDP Theorem. In *Logic and Algorithmic*, volume 30 of *Monographies de L'Enseignement Mathématique*, pages 187–206. Université de Genève, Geneva, 1982.

[10] Petr Hájek and Pavel Pudlák. *Metamathematics of First-Order Arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1993.

[11] Richard Kaye. Model-theoretic properties characterizing Peano arithmetic. *The Journal of Symbolic Logic*, 56(3):949–963, September 1991.

[12] Richard Kaye. *Models of Peano Arithmetic*, volume 15 of *Oxford Logic Guides*. Clarendon Press, Oxford, 1991.

[13] Leszek Pacholski, Jedrzej Wierzejewski, and Alec J. Wilkie, editors. *Model Theory of Algebra and Arithmetic*, volume 834 of *Lecture Notes in Mathematics*, Berlin, 1980. Springer-Verlag. Proceedings of the Conference on Applications of Logic to Algebra and Arithmetic Held at Karpacz, Poland, September 1–7, 1979.

[14] Jeff B. Paris. A hierarchy of cuts in models of arithmetic. In Pacholski et al. [13], pages 312–337.

[15] Jeff B. Paris. Some conservation results for fragments of arithmetic. In Chantal Berline, Kenneth Mc Aloon, and Jean-Pierre Ressayre, editors. *Model Theory and Arithmetic*, volume 890 of *Lecture Notes in Mathematics*, pages 251–262. Springer-Verlag, Berlin, 1981.

[16] Jeff B. Paris and Laurence A. S. Kirby. Σ_n-collection schemas in arithmetic. In Angus Macintyre, Leszek Pacholski, and Jeff B. Paris, editors, *Logic Colloquium '77*, volume 96 of *Studies in Logic and the Foundations of Mathematics*, pages 199–209. North-Holland Publishing Company, Amsterdam, 1978.

[17] Craig Smoryński. Cofinal extensions of nonstandard models of arithmetic. *Notre Dame Journal of Formal Logic*, 22(2):133–144, April 1981.

[18] Alfred Tarski and Robert L. Vaught. Arithmetical extensions of relational systems. *Compositio Mathematica*, 13:81–102, 1956–1958.

[19] Alex J. Wilkie. Applications of complexity theory to Δ_0-definability problems in arithmetic. In Pacholski et al. [13], pages 363–369.

[20] Alex J. Wilkie and Jeff B. Paris. On the existence of end extensions of models of bounded induction. In Jens Erik Fenstad, Ivan T. Frolov, and Risto Hilpinen, editors. *Logic, Methodology and Philosophy of Science VIII*, volume 126 of *Studies in Logic and the Foundations of Mathematics*, pages 143–161. North-Holland Publishing Company, Amsterdam, 1989.