MODEL THEORY OF ARITHMETIC

Lecture 9: The Mac Dowell–Specker Theorem

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> Einem Modell M von [PA] ist in natürlicher Weise eine additive Struktur G_M zugeordnet: G_M ist die additive Gruppe von M. [...] Unser Ziel ist die Charakterisierung der Gruppen G, welche in der Rolle als Gruppen G_M auftreten.
>
> Mac Dowell and Specker [6]

By Remark 8.7 and Theorem 7.4, whenever $n \in \mathbb{N}$ and M is a countable (nonstandard) model of PA, we can find a proper n-elementary end extension of M. In fact, an omitting types argument shows that every countable model of PA has a proper elementary end extension. The types one needs to omit when end extending a model $M \models \mathrm{PA}$ are those of the form

$$p_b(v) = \{v < b\} \cup \{v \neq a : a \in M\},$$

where $b \in M$. If M is countable, then provided we can make each p_b non-isolated, we can omit all of them in a single model by the Omitting Types Theorem. Surprisingly, this countability condition is actually not necessary, although one cannot omit uncountably many non-principal types simultaneously in general.

Mac Dowell–Specker Theorem [6]. Every model of PA has a proper elementary end extension.

Remark 9.1. The assumption that the ground model satisfies PA is necessary by Theorem 6.3.

Remark 9.2. The analogous question about cofinal extensions (of nonstandard models) has an easy answer by the Splitting Theorem.

The reason why we can omit uncountably many non-isolated types in this situation is that the least number principle provides a *definable* witness to every true existential statement. This gives us models which omit *all* non-isolated types, however many there are. These models are called *atomic*. In fact, we will get models that are *prime*, but primeness is not needed for this lecture.

Proposition 9.3. Let $\mathscr{L}_{\mathrm{A}}^* \supseteq \mathscr{L}_{\mathrm{A}}$, and $T \supseteq \mathrm{PA}^-$ that is complete, consistent as an $\mathscr{L}_{\mathrm{A}}^*$-theory, and includes the induction axiom

$$\forall \bar{z}\ \big(\theta(0, \bar{z}) \wedge \forall x\ \big(\theta(x, \bar{z}) \rightarrow \theta(x+1, \bar{z})\big) \rightarrow \forall x\ \theta(x, \bar{z})\big)$$

for every $\theta \in \mathscr{L}_{\mathrm{A}}^*$. Then T has a unique prime model in which every element is

$$(\min x)(\eta(x))$$

for some $\eta \in \mathscr{L}_{\mathrm{A}}^*$.

Proof sketch. Notice our proof of Theorem 2.3 actually works for all languages that extend $\mathscr{L}_{\mathrm{A}}$. So T has the least number principle for all $\mathscr{L}_{\mathrm{A}}^*$-formulas. Let

$$\hat{\mathscr{L}}_{\mathrm{A}}^* = \mathscr{L}_{\mathrm{A}}^* \cup \{c_\eta : T \vdash \exists x\ \eta(x), \text{ where } \eta \in \mathscr{L}_{\mathrm{A}}^*\},$$

where the c_η's are new constant symbols. Define the $\hat{\mathscr{L}}_{\mathrm{A}}^*$-theory $\hat{T}$ by

$$\hat{T} = T \cup \{\eta(c_\eta) \wedge \forall x{<}c_\eta\ \neg\eta(x) : T \vdash \exists x\ \eta(x)\}.$$

Then $\hat{T}$ is essentially just T with some new names for definable objects. Hence, since T is complete and consistent, so is $\hat{T}$. Moreover, the theory $\hat{T}$ is Henkinized, because the new symbols can all be replaced by their definitions. Let $\hat{K}$ be the term model of $\hat{T}$. Then the reduct K of $\hat{K}$ to $\mathscr{L}_{\mathrm{A}}^*$ is what we want. Notice for each $c \in K$, there is $\eta \in \mathscr{L}_{\mathrm{A}}^*$ such that

$$\mathrm{tp}_K(c) = \big\{\theta(v) \in \mathscr{L}_{\mathrm{A}}^* : T \vdash \forall v \ \big(\eta(v) \wedge \forall x{<}v \ \neg\eta(x) \rightarrow \theta(v)\big)\big\}. \qquad \square$$

Alternatively, one can prove this proposition by taking the definable closure (of the empty set) in any model of T.

The plan for getting an end extension is to add an 'ideal element' on top of the ground model. We control the type of this 'ideal element' so that it does not entail the existence of any new element below an old element. This reduces the problem of building an end extension to that of building a suitable type.

Definition. Let M be a structure for a language $\mathscr{L}$.

- Denote by $\mathrm{Def}(M)$ the set of parametrically definable sets in M.
- A *complete M-type* is a type $p(\bar{v})$ over M such that for all $\theta(\bar{v}) \in \mathscr{L}(M)$,

$$\text{either} \quad \theta(\bar{v}) \in p(\bar{v}) \quad \text{or} \quad \neg\theta(\bar{v}) \in p(\bar{v}).$$

- A complete M-type $p(\bar{v})$ is *definable* if for every $\varphi(\bar{v}, \bar{z}) \in \mathscr{L}$,

$$\{\bar{z} \in M : \varphi(\bar{v}, \bar{z}) \in p(\bar{v})\} \in \mathrm{Def}(M).$$

- An extension $K \supseteq M$ is *conservative* if for every $X \in \mathrm{Def}(K)$,

$$X \cap M \in \mathrm{Def}(M).$$

Remark 9.4. One sees that allowing parameters in the formula φ in the definition of definable types does not change the notion.

Intuitively speaking, a type is definable if and only if the ground model knows every slice of it in a definable way. Similarly, an extension is conservative if and only if the ground model knows every definable set in the extension in a definable way. It thus seems apparent that the two notions should be closely connected to each other. In fact, the extension obtained by adjoining an 'ideal element' to a model of PA is conservative if and only if the type of this 'ideal element' is definable.

Definition. Let $M \models \mathrm{PA}$ and $p(\bar{v})$ be a complete M-type. Then $M(p)$ denotes the $\mathscr{L}_{\mathrm{A}}$-reduct of the prime model of $p(\bar{d})$ as given by Proposition 9.3, where $\bar{d}$ are new constant symbols.

Such $M(p)$'s are elementary extensions of M because $p(\bar{v}) \supseteq \mathrm{ElemDiag}(M)$.

Proposition 9.5. Let $M \models \mathrm{PA}$. Then the following are equivalent for a complete M-type $p(\bar{v})$.

(a) p is a definable type.

(b) $M(p)$ is a conservative extension of M.

Proof. Let $\bar{d} \in M(p)$ realizing p such that $M(p)$ is the prime model of $p(\bar{d})$.

For (a) $\Rightarrow$ (b), suppose p is a definable type. Consider

$$X = \{\bar{z} \in M(p) : M(p) \models \theta(c, \bar{z})\},$$

where $\theta \in \mathscr{L}_{\mathrm{A}}$ and $c \in M(p)$. Find $\eta \in \mathscr{L}_{\mathrm{A}}(M)$ such that

$$c = (\min x)(\eta(x, \bar{d})).$$

This is possible because of the definition of $M(p)$. Then

$$\begin{aligned} X \cap M &= \{\bar{z} \in M : M(p) \models \theta(c, \bar{z})\} \\ &= \big\{\bar{z} \in M : M(p) \models \theta\big((\min x)(\eta(x, \bar{d})), \bar{z}\big)\big\} \\ &= \big\{\bar{z} \in M : \theta\big((\min x)(\eta(x, \bar{v})), \bar{z}\big) \in p(\bar{v})\big\} \in \mathrm{Def}(M) \end{aligned}$$

since p is a definable type, cf. Remark 9.4. So $M(p)$ is a conservative extension of M.

Conversely, suppose (b) holds. Now if $\varphi(\bar{v}, \bar{z}) \in \mathscr{L}_{\mathrm{A}}$, then

$$\{\bar{z} \in M : \varphi(\bar{v}, \bar{z}) \in p(\bar{v})\} = \{\bar{z} \in M : M(p) \models \varphi(\bar{d}, \bar{z})\} \in \mathrm{Def}(M)$$

by conservativity. Thus p is a definable type. □

Recall that we actually want end extensions.

Proposition 9.6. Every conservative $K \supseteq M \models \mathrm{PA}$ with $K \models \mathrm{PA}^-$ is an end extension.

Proof. Let $a \in K \setminus M$. Then by conservativity,

$$\{z \in M : z < a\} \in \mathrm{Def}(M).$$

This set contains 0 and is closed under successor. So it is M by induction. Hence $M < a$. □

In view of these two propositions, to show the Mac Dowell–Specker Theorem, it suffices to find a definable type over a given model of PA. This slight detour is, in a sense, inevitable, because there are models of PA all of whose elementary end extensions are conservative [5, Section 2.2.2]. It is not necessary to work within second-order arithmetic, but it is definitely nicer to do so.

Definition. ACA_0 is the $\mathscr{L}_{\mathrm{II}}$-theory axiomatized by

- the axioms of PA^-;
- the *induction axiom*

$$\forall X \ \big(0 \in X \wedge \forall x \ (x \in X \to x+1 \in X) \to \forall x \ (x \in X)\big);$$

- *arithmetical comprehension* (ACA): for every arithmetical formula θ,

$$\forall \bar{z}, \bar{Z} \ \exists X \ \forall x \ \big(x \in X \leftrightarrow \theta(x, \bar{z}, \bar{Z})\big).$$

Notice ACA_0 has an induction *axiom* rather than an induction *scheme*. The amount of comprehension thus determines how much induction it possesses. It follows that PA and ACA_0 have essentially the same amount of induction.

Proposition 9.7. (a) For all $\mathscr{L}_{\mathrm{A}}$-structures M,

$$(M, \mathrm{Def}(M)) \models \mathrm{ACA}_0 \quad \Leftrightarrow \quad M \models \mathrm{PA}.$$

(b) If $(M, \mathscr{X}) \models \mathrm{ACA}_0$, then $(M, A)_{A \in \mathscr{X}}$ satisfies full induction, where the A's are all considered as new predicates.

(c) $\mathrm{ACA}_0 \vdash \mathrm{WKL}_0$.

Proof sketch. (a) For the $\Rightarrow$ implication, notice every element of $\mathrm{Def}(M)$ can be replaced by the formula that defines it.

(b) Observe there is no set quantification in the first-order language for $(M, A)_{A \in \mathscr{X}}$.

(c) The leftmost unbounded branch in an unbounded binary tree is arithmetically definable over the tree. □

We employ some combinatorics from recursion theory to help us build definable types. Recall from Lemma 2.1 that every first-order object in a model of RCA_0 can be considered as the code of a pair of numbers. So every set R can be viewed as the code of a sequence of sets $(R)_0, (R)_1, \ldots$, where $(R)_x$ denotes 'the xth column' of R.

Definition. Let $(M, \mathscr{X}) \models \mathrm{RCA}_0$. For $R \in \mathscr{X}$ and $x \in M$, set

$$(R)_x = \{y \in M : \langle x, y \rangle \in R\}.$$

If $X, Y \subseteq M$, then $X \subseteq^* Y$ means $X \setminus Y \not\subseteq_{\mathrm{cf}} M$, and $X^{\mathrm{c}} = M \setminus X$.

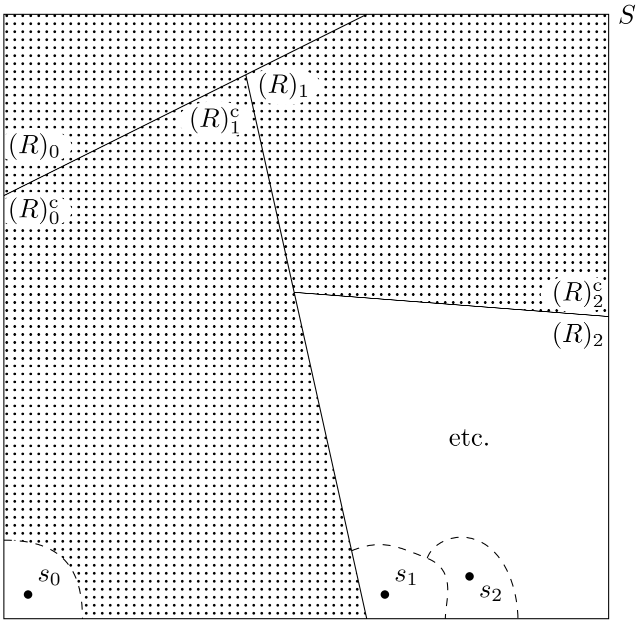

Figure 9.1: Constructing an R-cohesive subset of S

Lemma 9.8. $\mathrm{ACA}_0 \vdash \mathrm{COH}$, where COH says

for all unbounded S and all R, there exists $H \subseteq_{\mathrm{cf}} S$ such that for every z,

$$\text{either} \quad H \subseteq^* (R)_z \quad \text{or} \quad H \subseteq^* (R)_z^{\mathrm{c}}.$$

Such H is said to be *R-cohesive.* It is known [2] that $\mathrm{RCA}_0 + \mathrm{COH}$ is much weaker than ACA_0.

Proof sketch. Put $x \in H$ if and only if

there exists a sequence $s_0, s_1, \ldots, s_k$ such that $x = s_k$, and for all $j \leqslant k$, if

$$S_j = S \cap \bigcap\{(R)_i : i < j \wedge s_i \in (R)_i\} \cap \bigcap\{(R)_i^{\mathrm{c}} : i < j \wedge s_i \in (R)_i^{\mathrm{c}}\}$$

has unbounded intersection with $(R)_j$, then $s_j = \min\big((S_j \cap (R)_j) \setminus \{s_i : i < j\}\big)$, else $s_j = \min\big((S_j \cap (R)_j^{\mathrm{c}}) \setminus \{s_i : i < j\}\big)$.

□

Remark 9.9. It can be seen from the proof sketch above that given S and R, we can define an R-cohesive $H \subseteq S$ in a uniform way without having to refer to further parameters. In particular, if both S and R are parameter-free definable in M, then we can also make H so.

Theorem 9.10 (Gaifman [4], Phillips [8])**.** Every $M \models \mathrm{PA}$ admits a definable type $p(v)$ such that $M(p) \neq M$.

Proof. Work in $(M, \mathrm{Def}(M))$. We will find

$$M = S_0 \supseteq_{\mathrm{cf}} S_1 \supseteq_{\mathrm{cf}} S_2 \supseteq_{\mathrm{cf}} \cdots$$

in $\mathrm{Def}(M)$. Let $\theta_0(v), \theta_1(v), \theta_2(v), \ldots \in \mathscr{L}_{\mathrm{A}}(M)$ define $S_0, S_1, S_2, \ldots$ respectively. Our definable type $p(v)$ will be the deductive closure of

$$p_0(v) = \mathrm{ElemDiag}(M) + \{\theta_i(v) : i \in \mathbb{N}\} + \{v > a : a \in M\}.$$

This set is consistent by compactness. If p is complete, then clearly $M(p) \neq M$.

Suppose S_i is found. Consider $\varphi_i(v,z) \in \mathscr{L}_\mathrm{A}$, which comes from some fixed enumeration $(\varphi_i)_{i\in\mathbb{N}}$ of $\mathscr{L}_\mathrm{A}$-formulas. Notice we do not allow parameters to appear in φ here, because otherwise we may not be able to enumerate all such formulas in a countable sequence. Let

$$R_i = \{\langle z, v\rangle \in M : M \models \varphi_i(v,z)\} \in \mathrm{Def}(M).$$

Apply COH to find $S_{i+1} \subseteq_\mathrm{cf} S_i$ in $\mathrm{Def}(M)$ such that for all $z \in M$,

$$\text{either} \quad S_{i+1} \subseteq^* (R_i)_z \quad \text{or} \quad S_{i+1} \subseteq^* (R_i)_z^\mathrm{c}.$$

Now for each $z \in M$, the following chain of implications holds.

- $S_{i+1} \not\subseteq^* (R_i)_z^\mathrm{c} \Rightarrow S_{i+1} \subseteq^* (R_i)_z$ by the choice of S_{i+1}.
- $S_{i+1} \subseteq^* (R_i)_z \Rightarrow M \models \exists a\ \forall v{>}a\ \big(\theta_{i+1}(v) \to \varphi_i(v,z)\big)$.
- $M \models \exists a\ \forall v{>}a\ \big(\theta_{i+1}(v) \to \varphi_i(v,z)\big) \Rightarrow \varphi_i(v,z) \in p(v)$ by the definition of $p(v)$.
- $\varphi_i(v,z) \in p(v) \Rightarrow \neg\varphi_i(v,z) \notin p(v)$ since $p(v)$ is consistent.
- $\neg\varphi_i(v,z) \notin p(v) \Rightarrow M \not\models \exists a\ \forall v{>}a\ \big(\theta_{i+1}(v) \to \neg\varphi_i(v,z)\big)$.
- $M \not\models \exists a\ \forall v{>}a\ \big(\theta_{i+1}(v) \to \neg\varphi_i(v,z)\big) \Rightarrow S_{i+1} \not\subseteq^* (R_i)_z^\mathrm{c}$.

Therefore, all the clauses above are equivalent. This makes $p(v)$ complete because $\neg\varphi_i(v,z) \notin p(v)$ implies $\varphi_i(v,z) \in p(v)$ for all $z \in M$. It also says

$$\{z \in M : \varphi_i(v,z) \in p(v)\} = \big\{z \in M : M \models \exists a\ \forall v{>}a\ \big(\theta_{i+1}(v) \to \varphi_i(v,z)\big)\big\} \in \mathrm{Def}(M).$$

So $p(v)$ is a definable type. □

Remark 9.11. A careful examination of the proof above reveals that, in view of Remark 9.9, every θ_i can be made parameter-free. So essentially the same type works for all models of PA.

To prove the Mac Dowell–Specker Theorem, combine Proposition 9.5, Proposition 9.6, and Theorem 9.10.

Further exercises

There is currently no consensus about what an end extension of a model of *second-order* arithmetic means. Our definition here has the advantage of allowing the Mac Dowell–Specker Theorem to generalize naturally.

Definition. Let $(M,\mathscr{X}), (K,\mathscr{Y}) \models \mathrm{PA}^-$. We say that an embedding $i\colon (M,\mathscr{X}) \to (K,\mathscr{Y})$ is an *end extension* if $i(M) \subseteq_\mathrm{e} K$. It is *arithmetically elementary* if it is elementary for all arithmetical formulas. It is *proper* if $i(M) \neq K$.

Theorem 9.12. The following are equivalent for a countable $(M,\mathscr{X}) \models \mathrm{RCA}_0$.

(a) $(M,\mathscr{X}) \models \mathrm{ACA}_0$.

(b) $(M,\mathscr{X})$ has a proper arithmetically elementary end extension $i\colon (M,\mathscr{X}) \to (K,\mathscr{Y})$ such that $\mathscr{X} = \{X \cap M : X \in \mathrm{Def}(K)\}$.

Proof. Let $\mathscr{L}_\mathrm{A}^*$ be the language obtained from $\mathscr{L}_\mathrm{A}$ by adding a new unary predicate symbol for each element of $\mathscr{X}$. Then $M^* = (M,A)_{A\in\mathscr{X}}$ is an $\mathscr{L}_\mathrm{A}^*$-structure.
First suppose (a) holds.

(1) Imitate the proof of Theorem 9.10 to find a complete M^*-type $p(v)$ such that

  (i) $\{z \in M : \varphi(v,z) \in p(v)\} \in \mathscr{X}$ for every $\varphi \in \mathscr{L}_\mathrm{A}^*$; and

  (ii) $M^*(p) \neq M^*$, where $M^*(p)$ denotes the $\mathscr{L}_\mathrm{A}^*$-reduct of the prime model of $p(v)$ as given by Proposition 9.3.

Suppose we have such a type $p(v)$. Let K be the $\mathscr{L}_{\mathrm{A}}$-reduct of $M^*(p)$, and let $i(x) = x$ for every $x \in M$. Set $\mathscr{Y} = \{i(A) \subseteq K : A \in \mathscr{X}\}$, where $i(A)$ is the interpretation of the predicate symbol A in $M^*(p)$.

(2) Show that $i\colon (M, \mathscr{X}) \to (K, \mathscr{Y})$ is a proper arithmetically elementary end extension in which

$$\mathscr{X} \supseteq \{X \cap M : X \in \mathrm{Def}(K)\}.$$

Notice M^* satisfies full induction in the language $\mathscr{L}_{\mathrm{A}}^*$. The proof of Theorem 2.7 then generalizes to show that $S{\restriction}a \in \mathrm{Cod}(M)$ for every $S \in \mathrm{Def}(M^*)$ and every $a \in M$. This transfers to $M^*(p)$.

(3) Deduce that $\mathscr{X} \subseteq \{X \cap M : X \in \mathrm{Def}(K)\}$.

Conversely, suppose (b) holds, as witnessed by $i\colon (M, \mathscr{X}) \to (K, \mathscr{Y})$. It is a fact that the proof of Theorem 6.3 can be adapted to show that M^* satisfies full induction in $\mathscr{L}_{\mathrm{A}}^*$. This transfers to $K^* = (K, i(A))_{A \in \mathscr{X}}$.

(4) Using the aforementioned generalization of Theorem 2.7, or otherwise, show that (a) holds.

(5) Where was the countability of $(M, \mathscr{X})$ used? □

Further reading

Stability theory

Although the notions of definable types [4] and conservative extensions [8] originate from the model theory of arithmetic, they have now become central notions in stability theory. See Baldwin's book [1] for more information.

Further comments

End extensions elementary for bigger languages

The Further exercises show how the Mac Dowell–Specker Theorem generalizes to structures that satisfy full induction in a *countable* language extending $\mathscr{L}_{\mathrm{A}}$. For uncountable languages, such generalizations fail in general [7, 3].

References

[1] John T. Baldwin. *Fundamentals of Stability Theory.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1988.

[2] Damir Dzhafarov. Reverse Mathematics Zoo. Available at `http://rmzoo.math.uconn.edu/`.

[3] Ali Enayat. A standard model of Peano arithmetic with no conservative elementary extension. *Annals of Pure and Applied Logic*, 156(2–3):308–318, December 2008.

[4] Haim Gaifman. Models and types of Peano's arithmetic. *Annals of Mathematical Logic*, 9(3):223–306, March 1976.

[5] Roman Kossak and James H. Schmerl. *The Structure of Models of Peano Arithmetic*, volume 50 of *Oxford Logic Guides*. Clarendon Press, Oxford, 2006.

[6] R. Mac Dowell and E. Specker. Modelle der Arithmetik. In *Infinitistic Methods*, pages 257–263. Pergamon Press, Oxford, 1961.

[7] George Mills. A model of Peano arithmetic with no elementary end extension. *The Journal of Symbolic Logic*, 43(1):563–567, September 1978.

[8] Robert G. Phillips. Omitting types in arithmetic and conservative extensions. In Albert Hurd and Peter Loeb, editors. *Victoria Symposium on Nonstandard Analysis*, volume 369 of *Lecture Notes in Mathematics*, pages 195–202. Springer-Verlag, Berlin, 1974.